Lesson 5: More estimating probabilities

Goals

- Describe (orally and in writing) reasons why the relative frequency from an experiment may not exactly match the actual probability of the event.
- Recognise that sometimes the outcomes in a sample space are not equally likely.
- Use the results from a repeated experiment to estimate the probability of an event, and justify (orally and in writing) the estimate.

Learning Targets

- I can calculate the probability of an event when the outcomes in the sample space are not equally likely.
- I can explain why results from repeating an experiment may not exactly match the expected probability for an event.

Lesson Narrative

In this lesson students compare the results from running actual trials of an experiment to the expected, calculated probabilities. They also use their data to see that additional trials usually produce more accurate results as minor differences even out after many trials.

In the first activity, students spin four different spinners to see that the outcomes in a sample space may not be equally likely, and they examine the spinners to construct arguments about why some outcomes are more likely than others. In the next activity, students draw blocks out of a bag repeatedly and use the relative frequency to estimate the probability of getting a green block. This activity differs from the activity in the previous lesson where students were rolling a cube repeatedly because in this lesson the students do not know the probability of getting a green block before they start the experiment.

In future lessons students will be asked to design and use simulations. Each lesson leading up to that helps prepare students by giving them hands-on experience with different types of chance experiments they could choose to use in their simulations. In this lesson students work with spinners and drawing blocks out of a bag.

Addressing

- Understand that the probability of a chance event is a number between 0 and 1 that expresses the likelihood of the event occurring. Larger numbers indicate greater likelihood. A probability near 0 indicates an unlikely event, a probability around $\frac{1}{2}$ indicates an event that is neither unlikely nor likely, and a probability near 1 indicates a likely event.
- Approximate the probability of a chance event by collecting data on the chance process that produces it and observing its long-run relative frequency, and predict the

approximate relative frequency given the probability. For example, when rolling a dice 600 times, predict that a 3 or 6 would be rolled roughly 200 times, but probably not exactly 200 times.

- Develop a probability model and use it to find probabilities of events. Compare probabilities from a model to observed frequencies; if the agreement is not good, explain possible sources of the discrepancy.
- Develop a probability model (which may not be uniform) by observing frequencies in data generated from a chance process. For example, find the approximate probability that a spinning penny will land heads up or that a tossed paper cup will land open-end down. Do the outcomes for the spinning penny appear to be equally likely based on the observed frequencies?

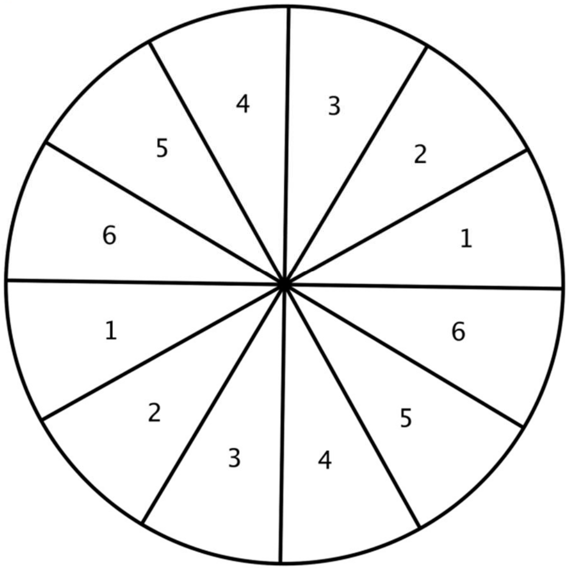
Instructional Routines

- Co-Craft Questions
- Discussion Supports

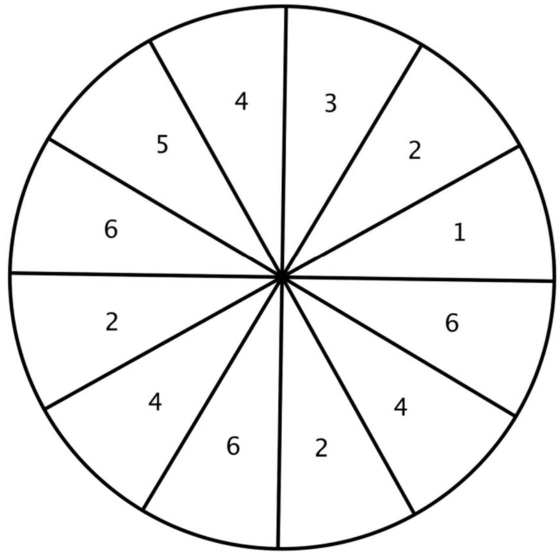
Required Materials

Copies of blackline master: Making My Head Spin

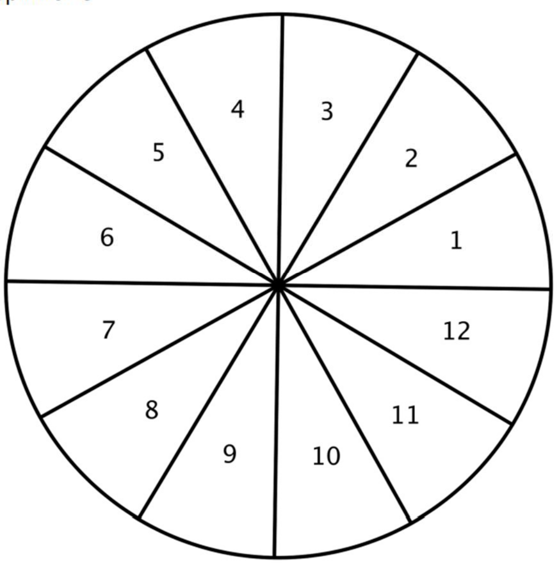
Spinner A



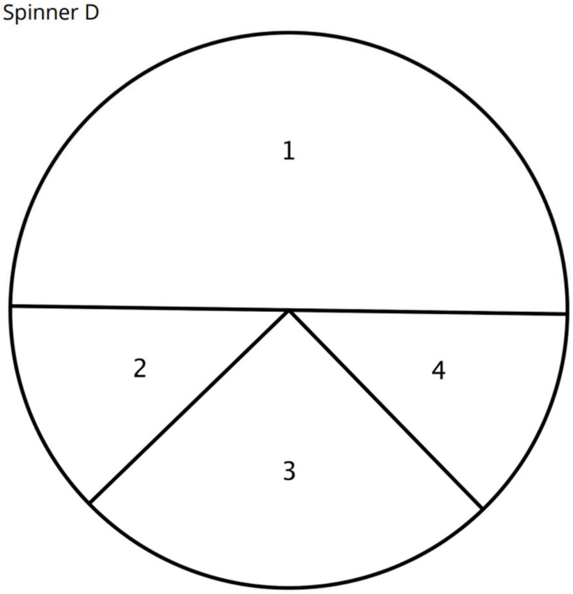
Spinner B



Spinner C



Spinner D



Paper bags
Paper clips
Multi-link cubes

Required Preparation

Provide 1 set of 4 spinners cut from the Making My Head Spin blackline master for every 4 students. Each student will need a pencil and paper clip to use with the spinners.

For the How Much Green activity, prepare a paper bag containing 5 multi-link cubes (3 green and 2 of another matching colour) for every 3–4 students.

Student Learning Goals

Let's estimate some probabilities.

5.1 Is it Likely?

Warm Up: 5 minutes

The purpose of this warm-up is for students to think more deeply about probabilities and what the values actually represent. In this activity, students are asked to compare the likelihood of three events with probabilities given in different formats. In the discussion, students are also asked to think about the context of the situations to see that probabilities are not the only consideration when planning a response.

Launch

Give students 2 minutes of quiet work time followed by a whole-class discussion.

Student Task Statement

1. If the weather forecast calls for a 20% chance of light rain tomorrow, would you say that it is likely to rain tomorrow?
2. If the probability of a tornado today is $\frac{1}{10}$, would you say that there will likely be a tornado today?
3. If the probability of snow this week is 0.85, would you say that it is likely to snow this week?

Student Response

1. It is not likely to rain tomorrow, but it could happen.
2. The tornado is not likely to happen today, but it could happen.
3. It is likely that it will snow this week, but it might not happen.

Activity Synthesis

Ask students, "Which situation would you worry about the most? Is that the same situation that is the most likely?"

Note that our interpretation of the scenario influences how we feel about how likely an event is to happen. Although the likelihood of rain is higher, the implications of a tornado are much greater, so you may be more likely to worry about the tornado than the rain.

5.2 Making My Head Spin

20 minutes (there is a digital version of this activity)

In this activity, students return to calculating probabilities using the sample space, and they compare the calculated probabilities to the outcomes of their actual trials. Students have a chance to construct arguments about why probability estimates based on carrying out the experiment many times might differ from the expected probability. Students use a spinner in this activity, which will be helpful when designing simulations in upcoming lessons.

Instructional Routines

- Discussion Supports

Launch

Arrange students in groups of 4. Provide 1 set of 4 spinners cut from the blackline master to each group. Each student will need a pencil and paper clip. Demonstrate how to use a pencil and paper clip to spin the spinner: Unbend one end of the paper clip so that it is straight. Put the paper clip on the end of the pencil and the pencil tip at the centre of the spinner. Spin the paper clip so that it rotates around the pencil and the unbent portion points to the result of the spin. If it is difficult to determine which section the end of the paper clip points to, it is okay to disregard that spin and spin again.

Classes using the digital curriculum have spinner applets to use if they choose to. These applets are based on the work of Terry Lee Lindenmuth in GeoGebra.

Following the teacher demonstration, give students 5 minutes of quiet work time, then 10 minutes of group work followed by a whole-class discussion.

Action and Expression: Internalise Executive Functions. Following the teacher demonstration, check for understanding by inviting students to rephrase directions in their own words.
Supports accessibility for: Memory; Conceptual processing

Anticipated Misconceptions

Students may think they need to have their probability estimates match the calculated probability. Remind them of the activity "Due For a Win" to see why we might expect the estimated probability and the calculated probability to be a little different.

Student Task Statement

Your teacher will give you 4 spinners. Make sure each person in your group uses a different spinner.

1. Spin your spinner 10 times, and record your outcomes.

2. Did you get all of the different possible outcomes in your 10 spins?

3. What fraction of your 10 spins landed on 3?

4. Next, share your outcomes with your group, and record their outcomes.

   a. Outcomes for spinner A:

   b. Outcomes for spinner B:

   c. Outcomes for spinner C:

   d. Outcomes for spinner D:

5. Do any of the spinners have the same sample space? If so, do they have the same probabilities for each number?

6. For each spinner, what is the probability that it lands on the number 3? Explain or show your reasoning.

7. For each spinner, what is the probability that it lands on something other than the number 3? Explain or show your reasoning.

8. Noah put spinner D on top of his closed binder and spun it 10 times. It never landed on the number 1. How might you explain why this happened?

9. Han put spinner C on the floor and spun it 10 times. It never landed on the number 3, so he says that the probability of getting a 3 is 0. How might you explain why this happened?

Student Response

1. Answers vary.

2. Answers vary.

3. Answers vary.

4. Answers vary.

5. Yes, spinners A and B have the same sample space. They do not have the same probabilities for the numbers, though. For example, spinner A has two 3s out of 12 equal spaces, so the probability of spinning a 3 is $\frac{2}{12}$. Spinner B only has one 3 out of 12 equal spaces, so the probability of spinning a 3 is $\frac{1}{12}$.

6. Spinner A: $\frac{1}{6}$. Spinner B: $\frac{1}{12}$. Spinner C: $\frac{1}{12}$. Spinner D: $\frac{1}{4}$. The values for spinners A, B, and C are calculated by counting the number of 3s on the spinner and dividing by the number of equal sections on the spinner. For spinner B, the section for 3 is one fourth of the circle.

7. Spinner A: $\frac{5}{6}$. Spinner B: $\frac{11}{12}$. Spinner C: $\frac{11}{12}$. Spinner D: $\frac{3}{4}$. Each of these values was calculated by counting the number of things that were not 3 on the spinner and dividing by the number of sections on the spinner.

8. Since the binder is sloped, gravity may have pulled the spinner so that a 1 would not show up.

9. Han might have been holding the spinner at an angle like Noah or maybe he just did not spin enough times. Since it is possible to spin a 1, the probability should not be 0.

Are You Ready for More?

Design a spinner that has a $\frac{2}{3}$ probability of landing on the number 3. Explain how you could precisely draw this spinner.

Student Response

Answers vary. Sample response: First, I would draw a circle with a compass. Then I would divide the circle into 3 equal sections by using a protractor and measuring an angle of $120°$ since $360 \div 3 = 120$. I would write the number 3 in two of the sections and write the number 1 in the other section.

Activity Synthesis

The purpose of this discussion is to think about reasons why the estimate of a probability may be different from the actual probability.

Select some students to share their responses to the last 5 questions.

Ask, "How does your fraction of 3 spins compare to the probability you expect from just looking at the spinner?"

Explain that although the spinners provided were designed to have equally sized sections (except for spinner D which has the angles $180°$, $45°$, and $90°$), sometimes it may be difficult to determine when the sections are exactly the same size. For some situations where things are not so evenly divided, some experimenting may be needed to determine that the outcomes actually follow the probability we might expect.

There are two main reasons why the fraction of the time an event occurs may differ from the actual probability:

- The simulation was designed or run poorly.
    - Maybe the spinner sections were not equal sized when they should be or maybe the spinner was tilted.
    - Maybe the items being chosen from the bag were different sizes, so you were more likely to grab one than another.

  - Maybe the coin or dice are not evenly weighted and are more likely to land with one side up than others.
  - Maybe the calculator that was programmed to return a random number has a problem with the code that returns some numbers more often than others.

- Not enough trials were run.
  - As in the previous lesson, if you flip a coin once and it comes up heads, that doesn't mean that it always will. Even if you flip it 100 times, it's not guaranteed to land heads up exactly 50 times, so some slight deviation is to be expected.

Speaking: Discussion Supports. To help students think about reasons why the frequency of spinning a number may be different from the actual probability of spinning that number, provide sentence frames such as: "When we spun the spinner ten times, we noticed . . .", "The probability of the spinner landing on the number three is _____ because . . .", and "The result of our experiment is similar to or different than the actual probability because . . . ." This will support a rich and inclusive discussion about why it is reasonable to see outcomes that are slightly different than the actual probability of the situation.
Design Principle(s): Support sense-making; Cultivate conversation

5.3 How Much Green?

10 minutes

In the previous activity, students could see the entire spinner and calculate the probability to compare with their experimental results. In many contexts, it is not possible to know the entire sample space or calculate an exact probability from the situation itself. For example, the probability of rain tomorrow usually cannot be exactly estimated from available information. In this activity, students see how to approach such problems by estimating the probability of an event using the results from repeating trials. In this particular example, the exact probability can be calculated when the information is revealed, so students can compare their results to this value. In situations like predicting the weather, estimates may be the best thing we have available. Students gain exposure to the process of drawing blocks from a bag, which will be useful in designing simulations in future lessons.

Instructional Routines

- Co-Craft Questions

Launch

Arrange students in groups of 3–4. Distribute 1 paper bag containing 5 multi-link cubes (3 green cubes and 2 cubes of some other colour that match each other) to each group. 5 minutes of group work followed by a whole-class discussion.

Conversing: Co-Craft Questions. Before presenting the questions in this activity, provide the bag of blocks and the instructions for the experiment. Ask students to write possible mathematical questions about the situation. Then ask students to compare the questions

they generated with a partner before sharing questions with the whole class. Listen for and amplify questions about estimating the probability of taking out a green block from the bag. Then reveal and ask students to work on the actual questions of the task. This will help develop students' meta-awareness of the language used to generate questions about the probability of an event.
Design Principle(s): Maximise meta-awareness

Anticipated Misconceptions

Some students may estimate a probability that is different from the fraction of times they draw a green block. Ask these students for a reason they chose a different value for their estimate.

Student Task Statement

Your teacher will give you a bag of blocks that are different colours. Do not look into the bag or take out more than 1 block at a time. Repeat these steps until everyone in your group has had 4 turns.

- Take one block out of the bag and record whether or not it is green.
- Put the block back into the bag, and shake the bag to mix up the blocks.
- Pass the bag to the next person in the group.

1. What do you think is the probability of taking out a green block from this bag? Explain or show your reasoning.
2. How could you get a better estimate without opening the bag?

Student Response

1. Answers vary. Sample response: I think the probability is $\frac{7}{12}$ since we got 7 green blocks after 12 trials.
2. Answers vary. Sample response: Continuing to pick out blocks more times might get a better estimate.

Activity Synthesis

The purpose of the discussion is to show that estimating the probability of an event can be done using repeated trials and is usually improved by including more trials.

Ask each group how many green blocks they got in their trials and display the class results for all to see.

Consider asking these discussion questions:

- "How can we use the values from the class to estimate the probability of drawing out a green block?" (By using the data we have, we can estimate the fraction of blocks that are green.)

- "Based on the class data, what is the estimated probability of choosing a green block from the bag?"
- "Was the probability estimated from the class data different from the probability estimate based on the data just from your group? Why?" (Yes, since not everyone picked out the same thing each time.)
- "Some of you may have felt that there are 5 blocks in the bag. If we use that information, does that change our estimate of the probability?" (If there are only 5 blocks, it only makes sense for the probability to be $0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$, or $\frac{5}{5}$.)
- Allow students to open the bags and see what blocks are in there. "What is the probability based on this observation?" $\left(\frac{3}{5}\right)$
- "Does it match your group estimates? Does it match the class estimates?"
- "Was the estimate from the class data more accurate than the estimates from the groups?" (It should be since more trials are included.)

Lesson Synthesis

Consider asking these discussion questions:

- "A student repeats the process of taking blocks out of a bag and replacing them 100 times. A green block is drawn 67 times. What is a good estimate for the probability of drawing out a green block from the bag?" $\left(\frac{67}{100}\right)$
- "A chance experiment is done a few times and the fraction of outcomes in a certain event is used as an estimate for the probability of the event. If the experiments are done carefully, how could the estimate be improved?" (Usually, the more trials done for an experiment, the closer the estimate will be to a calculated probability.)
- "A chance experiment is repeated many times, but the fraction of outcomes for which a certain event occurs does not match the actual probability of the event. What are some reasons this may happen?" (The experiment may not have been repeated enough times. The experiment was badly done. For example, the experiment may not have been as random as originally thought. We usually expect a little difference between the estimated probability and the actual probability.)

5.4 The Probability of Spinning B

Cool Down: 5 minutes

Student Task Statement

Jada, Diego, and Elena each use the same spinner that has four (not necessarily equal sized) sections marked A, B, C, and D.

- Jada says, "The probability of spinning B is 0.3 because I spun 10 times, and it landed on B 3 times."
- Diego says, "The probability of spinning B is 20% because I spun 5 times, and it landed on B once."
- Elena says, "The probability of spinning B is $\frac{2}{7}$ because I spun 7 times, and it landed on B twice."

1. Based on their methods, which probability estimate do you think is the most accurate? Explain your reasoning.
2. Andre measures the spinner and finds that the B section takes up $\frac{1}{4}$ of the circle. Explain why none of the methods match this probability exactly.

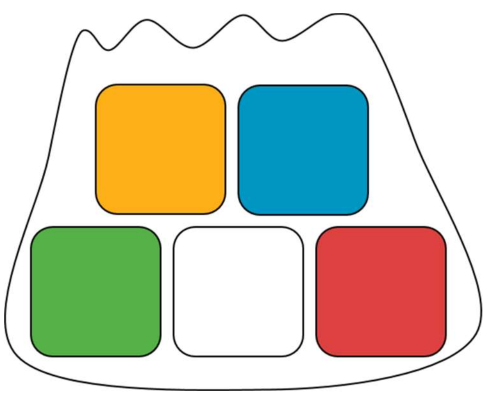
Student Response

Answers vary. Sample response:

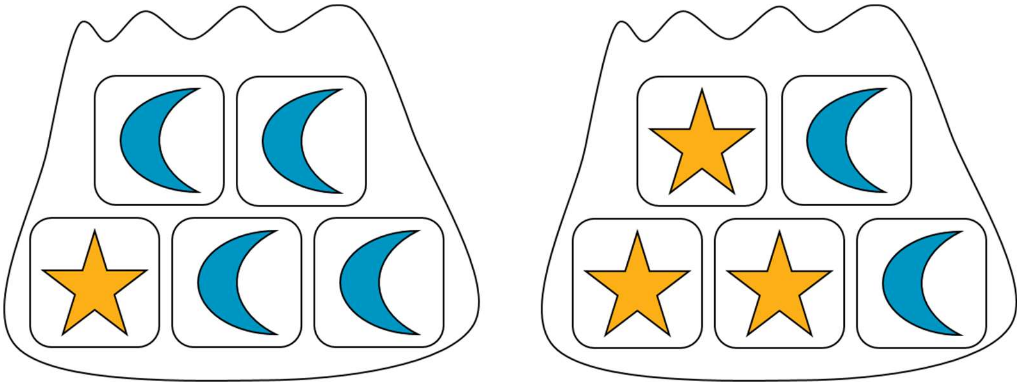
1. Jada's method is probably the most accurate since she had the most attempts.
2. Since Jada spun it 10 times, she could only get estimates in 0.1 increments. Since Diego spun it 5 times, he could only get estimates in 20% increments. Since Elena spun it 7 times, she could only get estimates in $\frac{1}{7}$ increments. If they spun the spinner more times, their results would probably get closer to $\frac{1}{4}$.

Student Lesson Summary

Suppose a bag contains 5 blocks. If we select a block at random from the bag, then the probability of getting any one of the blocks is $\frac{1}{5}$.



Now suppose a bag contains 5 blocks. Some of the blocks have a star, and some have a moon. If we select a block from the bag, then we will either get a star block or a moon block. The probability of getting a star block depends on how many there are in the bag.



In this example, the probability of selecting a star block at random from the first bag is $\frac{1}{5}$, because it contains only 1 star block. (The probability of getting a moon block is $\frac{4}{5}$.) The probability of selecting a star block at random from the second bag is $\frac{3}{5}$, because it contains 3 star blocks. (The probability of getting a moon block from this bag is $\frac{2}{5}$.)

This shows that two experiments can have the same sample space, but different probabilities for each outcome.

Lesson 5 Practice Problems

Problem 1 Statement

What is the same about these two experiments? What is different?

- Selecting a letter at random from the word "ALABAMA"
- Selecting a letter at random from the word "LAMB"

Solution

Answers vary. Sample response: Both these experiments have the same sample space. Also, they are both chance experiments that have to do with selecting letters at random from words. These two experiments are different because in the word "LAMB," each letter is equally likely, but in the word "ALABAMA," the letter "A" is more likely than the other letters.

Problem 2 Statement

Andre picks a block out of a bag 60 times and notes that 43 of them were green.

a. What should Andre estimate for the probability of picking out a green block from this bag?

b. Mai looks in the bag and sees that there are 6 blocks in the bag. Should Andre change his estimate based on this information? If so, what should the new estimate be? If not, explain your reasoning.

Solution

a. $\frac{43}{60}$

b. Yes. The estimate should be changed to $\frac{4}{6}$ since the original estimate is close to $\frac{40}{60}$, which is equal to $\frac{4}{6}$, and which is actually possible with 6 blocks. Since Andre was doing an experiment, it makes sense that he would be close to, but not exactly match the calculated probability.

Problem 3 Statement

Han has a dice that he suspects is not so standard.

– Han rolls the dice 100 times, and it lands on a six 40 times.

– Kiran rolls the dice 50 times, and it lands on a six 21 times.

– Lin rolls the dice 30 times, and it lands on a six 11 times.

Based on these results, is there evidence to help prove that this dice is not a standard dice? Explain your reasoning.

Solution

Yes. Sample explanation: A standard dice should land on a six about $\frac{1}{6}$ of the time. After 100 rolls, it should land on six about 16 or 17 times. All three people had it land on six more than twice as often. With this many rolls, there is strong evidence that this dice is not standard.

Problem 4 Statement

A textbook has 428 pages numbered in order starting with 1. You flip to a random page in the book in a way that it is equally likely to stop at any of the pages.

a. What is the sample space for this experiment?

b. What is the probability that you turn to page 45?

c. What is the probability that you turn to an even numbered page?

d. If you repeat this experiment 50 times, about how many times do you expect you will turn to an even numbered page?

Solution

a. The numbers 1 through 428

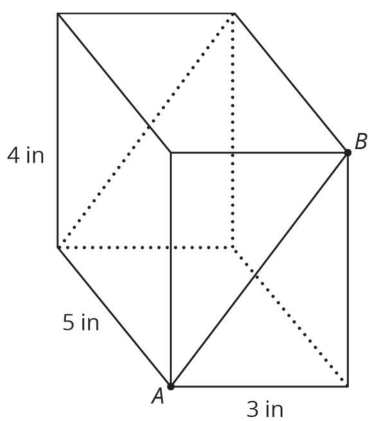
b. $\frac{1}{428}$

c. $\frac{214}{428}$ or $\frac{1}{2}$ (or equivalent)

d. About 25 times, because $\frac{1}{2} \times 50 = 25$.

Problem 5 Statement

A cuboid is cut along a diagonal on each face to create two triangular prisms. The distance between A and B is 5 inches.



What is the surface area of the original cuboid? What is the total surface area of the two triangular prisms together?

Solution

Cuboid: 94 square inches. Two triangular prisms together: 144 square inches