Lesson 1: Relationships of angles

Goals

- Comprehend and use the word “degrees” (in spoken and written language) and the symbol ∘ (in written language) to refer to the amount of turn between two different directions.
- Recognise 180° and 360° angles, and identify when adjacent angles add up to these amounts.
- Use reasoning about adjacent angles to determine the sizes of angles of pattern blocks, and justify (orally) the reasoning.

Learning Targets

- I can find the size of unknown angles by reasoning about adjacent angles with known sizes.
- I can recognise when an angle measures 90°, 180°, or 360°.

Lesson Narrative

Students were introduced to angles in KS2, when they drew angles, measured angles, identified angles as acute, right, or obtuse, and worked with adding and subtracting angles. Earlier in KS3, students also touched on angles briefly in their work with scale drawings. Now they begin a more detailed study of angles.

In this lesson, students gain hands-on experience composing, decomposing, and measuring angles. They refresh their memory about the relationship between **right angles**, **straight angles** (180°), and “all the way around” angles (360°), and they fit pattern blocks around a point to find out the angles at their vertices. They use simple equations they learned about in the previous unit to solve for angles.

Building On

- Measure angles in whole-number degrees using a protractor. Sketch angles of specified size.
- Recognise angle size as additive. Solve addition and subtraction problems to find unknown angles on a diagram in real world and mathematical problems, e.g., by using an equation with a symbol for the size of an unknown angle.

Addressing

- Draw, construct, and describe geometrical shapes and describe the relationships between them.
- Solve real-life and mathematical problems involving the size of angles, area, surface area, and volume.

Building Towards

- Solve real-life and mathematical problems involving angle size, area, surface area, and volume.
- Use facts about supplementary, complementary, vertical, and adjacent angles in a multi-step problem to write and solve simple equations for an unknown angle in a shape.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- Stronger and Clearer Each Time
- Collect and Display
- Co-Craft Questions
- Think Pair Share

Required Materials

Blank paper
Pattern blocks
Protractors
Clear protractors with no holes and with radial lines printed on them are recommended.

Scissors
Straightedges
A rigid edge that can be used for drawing line segments. Sometimes a ruler is okay to use as a straightedge, but sometimes it is preferable to use an unruled straightedge, like a blank index card.

Required Preparation

Prepare one set of pattern blocks for each group of 3–4 students, include blocks consisting of at least 3 yellow hexagons and 6 of each of the other shapes.

Student Learning Goals

Let’s examine some special angles.

1.1 Visualising Angles

Warm Up: 5 minutes (there is a digital version of this activity)

The purpose of this warm-up is to bring back to mind what students have learned previously about angle sizes, as well as to discuss what aspects of each shape is important and which aspects can be ignored. Students may benefit from the use of an Angle Window: a scrap of paper with a penny-sized hole torn in the centre of it. Students position the window so that the vertex of the angle and the beginning of the two rays are visible through the hole. This helps block out distractions, such as the lengths on the sides of the angle or other objects in the diagram.

The first question addresses the misconception that the size of an angle is related to lengths of line segments. The second question shows students they must be specific about how they refer to angles that share a vertex and introduces students to thinking about overlapping angles. Monitor for students who use different names for the same angle.

Launch

Give students 1 minute of quiet work time, followed by a whole-class discussion.

If using the digital activity, make sure students realise they can drag the angles to compare size.

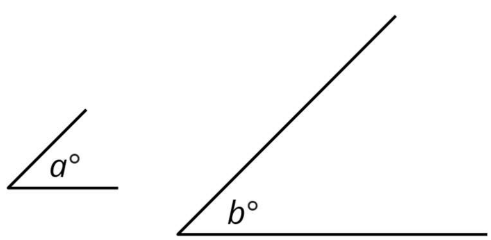
Anticipated Misconceptions

In the first question, students may say that the angle measuring b degrees is larger than the angle measuring a degrees because the line segments are longer. Show them how to use an Angle Window positioned over the vertex to focus on the amount of turn between the two rays and ignore the length of the line segments.

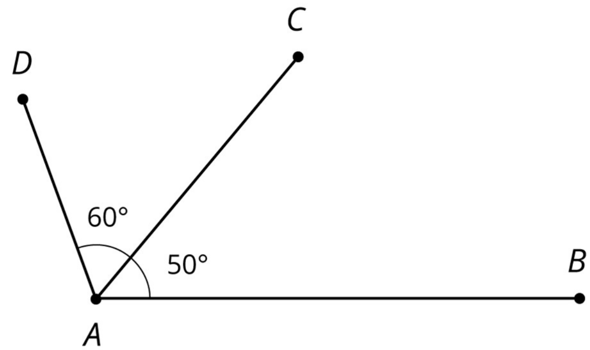
In the second question, students may say that there is no obtuse angle, because they are only looking at $\angle DAC$ and $\angle CAB$ and not noticing the overlapping angle $\angle DAB$. Reassure them that there is an obtuse angle in the shape, and ask them if $\angle DAC$ and $\angle CAB$ are the only angles present in the shape. Another possibility is to tell them that the obtuse angle measures 110 degrees to help them find it.

Student Task Statement

1. Which angle is bigger?



2. Identify an obtuse angle in the diagram.



Student Response

1. Neither. Both angles are the same size.

2. Angle DAB (or angle BAD) is obtuse. It measures 110° because $60 + 50 = 110$.

Activity Synthesis

The goal of this discussion is to ensure that students understand that angles measure the amount of turn between two different directions. Poll the class on their responses for the first question. Make sure students reach an agreement that both angles in the first question are the same size. If there is a lot of disagreement, it may be helpful to demonstrate the use of an Angle Window for the whole class. If using the digital version of the materials, either angle a or b can be dragged on top of the other to demonstrate that they have the same size.

Display the shape in the second question, and ask previously identified students to share their responses. Make sure students understand that saying angle A is not specific enough when referring to this diagram, because there is more than one angle with its vertex at point A. Consider asking questions like these:

- "What is the size of angle A?"
- "Which angle is angle A?"
- "Why is it not good enough to say angle A when referring to this diagram?"

Explain to the students that by using three points to refer to an angle, we can be sure that others will understand which angle we are talking about. Have students practise this way of referring to angles by asking questions such as:

- "Which angle is bigger, angle DAC or angle CAB?" (Angle DAC is bigger because it is 60 degrees. It doesn't matter that segment BA is longer than segment DA.)
- "Which angle is bigger, angle CAB or angle BAC?" (They are both the same size, because they are two names for the same angle.)

Also explain to students that in a diagram an arc is often placed between the two sides of the angle being referenced.

Tell students that angles DAC and CAB are known as **adjacent angles** because they are next to each other, sharing line segment AC as one of their sides and A as their vertex.

1.2 Pattern Block Angles

15 minutes (there is a digital version of this activity)

The purpose of this activity is to use the fact that the sum of the angles all the way around a point is 360° to reason about the size of other angles. Students are reminded that angle sizes are additive before undertaking work with complementary and supplementary angles in future lessons.

Formally, a right angle is 90° because we defined 360° to be all the way around and $\frac{1}{4} \times 360 = 90$. Students may have forgotten about 360°, but they are likely to remember 90° from their work with angles in KS2. We can use right angles as a tool to rediscover that all the way around must be 360°, because $4 \times 90 = 360$.

In this activity, students use pattern blocks to explore configurations that make 360° and to solve for angles of the individual blocks. For this activity, there are multiple configurations of blocks that will accomplish the task.

As students work, monitor for those who:

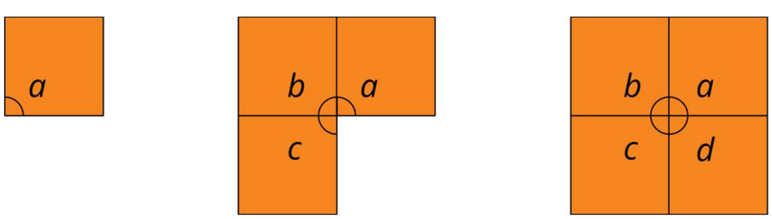
- use similar reasoning in the launch to figure out the size of the various angles they traced from the pattern blocks.
- find relationships between different angle sizes and different pattern blocks (for example: one hexagon angle is also 2 green triangles, which means one green triangle angle is 60° because $\frac{1}{2} \times 120 = 60$.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- Collect and Display

Launch

Arrange students in groups of 3–4. Display the shapes in this image one at a time, or use actual pattern blocks to recreate these shapes for all to see.



Ask these questions after each shape is displayed:

- "What is the size of $\angle a$? How do you know?" (90°, because it is a right angle.)
- "What is the size of $a + b + c$?" (270°, because $90 + 90 + 90 = 270$.)
- "What is the size of $a + b + c + d$?" (360°, because $4 \times 90 = 360$.)

Reinforce that 360° is once completely around a point by having students stand up, hold their arm out in front of them, and turn 360° around. Students who are familiar with activities like skateboarding or figure skating will already have a notion of 360° as a full rotation and 180° as half of a rotation.

Distribute pattern blocks. Or, if using the digital version of materials, demonstrate the use of the applet. Ensure students know that after they drag a block from the left to the right side of the window, they can click to rotate the block.

Representation: Develop Language and Symbols. Display or provide charts with the shapes, symbols and meanings of the angle sizes at the vertices for all the different pattern blocks. *Supports accessibility for: Conceptual processing; Memory*

Anticipated Misconceptions

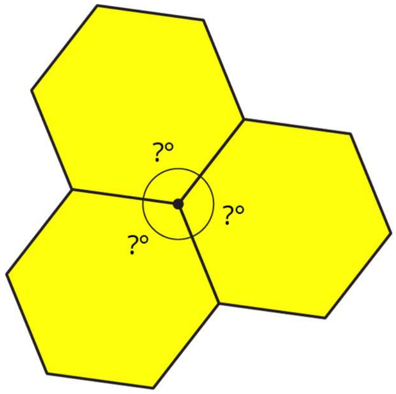
When working on calculating the size of the angle, students might need to be reminded that a complete turn is 360°.

If students place angles that are not congruent next to each other, it could produce valid reasoning, but they may draw erroneous conclusions. For example, using four copies of the blue rhombus, you can place 2 obtuse angles and 2 acute angles around the same vertex with no gaps or overlaps. However, this does not mean that they are each $\frac{1}{4}$ of 360°. Encourage students to reason about whether their conclusions make sense and to verify their conclusions in more than one way.

Student Task Statement

1. Trace one copy of every different pattern block. Each block contains either 1 or 2 angles with different sizes. Which blocks have only 1 unique angle? Which have 2?

2. If you trace three copies of the hexagon so that one vertex from each hexagon touches the same point, as shown, they fit together without any gaps or overlaps. Use this to figure out the size in degrees of the angle inside the hexagon pattern block.



3. Figure out the number of degrees in all of the other angles inside the pattern blocks that you traced in the first question. Be prepared to explain your reasoning.

Student Response

1. The hexagon, triangle, and square are all blocks with one unique size of angle. The trapezium and both rhombuses are blocks with two different sizes of angle.

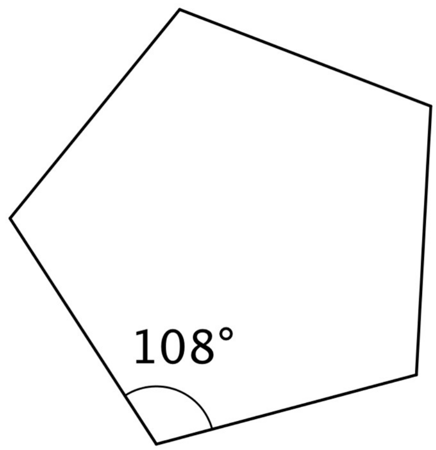
2. The size of the angle inside the yellow hexagon is 120°, since it takes 3 to go around a point and $360 \div 3 = 120$.

3. For the green triangle, all 3 angles measure 60°, since it takes 6 to go around a point and $360 \div 6 = 60$.
   For the tan rhombus, two of the angles measure 30°, since it takes 2 of them to equal the size of the angle in an equilateral triangle and $60 \div 2 = 30$. The other two angles measure 150°, since 5 of the smaller angles can fit together to equal this angle and $30 \times 5 = 150$.
   For the blue rhombus, two of the angles measure 60°, since they are the same angle as in the triangles, and the other two angles measure 120° since they are the same angle as in the hexagons.
   For the red trapezium, two of the angles measure 60° like the triangles, and the other two angles measure 120° like the hexagons.
   For the orange square, all 4 angles measure 90°, since it takes 4 angles to go around a point and $360 \div 4 = 90$.

Are You Ready for More?

We saw that it is possible to fit three copies of a regular hexagon snugly around a point.

Each interior angle of a regular pentagon measures 108°. Is it possible to fit copies of a regular pentagon snugly around a point? If yes, how many copies does it take? If not, why not?



Student Response

No. Three copies gives 324°, because $3 \times 108 = 324$. This is not enough—there would be a gap left over—because 360° is needed to get all the way around. Four copies gives 432°, because $4 \times 108 = 432$. This is too much! The fourth copy would overlap the first, not fit snugly.

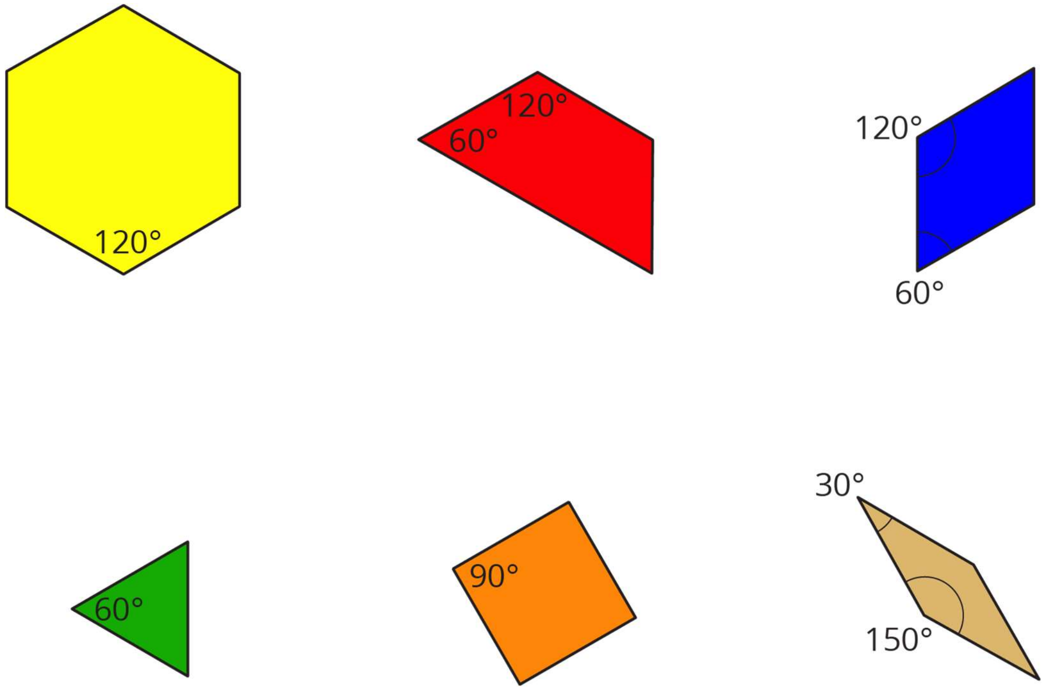
Activity Synthesis

The goal of this discussion is for students to be exposed to writing equations that represent the relationships between different sizes of angle. Select previously identified students to share how they figured out the different sizes of angle in each pattern block. Sequence the explanations from most common (reminiscent of the square and hexagon examples) to most creative.

Write an equation to represent how their angles add up to 360°. Listen carefully for how students describe their reasoning and make your equation match the vocabulary they use. For example, students might have reasoned about 6 green triangles by thinking $60 + 60 + 60 + 60 + 60 + 60 = 360$ or $6 \times 60 = 360$ or $360 \div 6 = 60$.

Once an angle from one block is known, it can be used to help figure out angles for other blocks. For example, students may say that they knew the angles on the yellow hexagon measured 120° because they could fit two of the green triangles onto one corner of the hexagon, and $60 + 60 = 120$ or $2 \times 60 = 120$. There are many different ways students could have reasoned about the angles on each block, and it is okay if they didn't think back to 360° for every angle.

Before moving on to the next activity, ensure that students know the size of each interior angle of each shape in the set of pattern blocks. Display these sizes for all to see throughout the remainder of the lesson.



Speaking, Listening: Collect and Display. Use this routine to capture existing student language related to finding the size of a given angle. Circulate and listen to student talk during small-group and whole-class discussion. Record the words, phrases, drawings, and writing students use to explain the equations they wrote to represent the relationships between different angle sizes. Display the collected language for all to see, and invite students to borrow from, or add more language to the display throughout the remainder of the lesson. It is expected that students will be using informal language when they explain their reasoning at this point in the unit. Over the course of the unit, invite students to suggest revisions, and updates to the display as they develop new mathematical ideas and new language to communicate them.
Design Principle(s): Support sense-making; Maximise meta-awareness

1.3 More Pattern Block Angles

10 minutes (there is a digital version of this activity)

In this activity, students figure out the sizes of given angles using the pattern block angles they discovered in the previous activity. Most importantly, students recognise that a straight angle can be considered an angle and not just a line. Students are asked to find different combinations of pattern blocks that form a straight angle, which helps students to see the connection between the algebraic action of summing angles and the geometric action of joining angles with the same vertex.

As students work on the task, monitor for students who use different combinations of blocks to form a straight angle.

Instructional Routines

- Co-Craft Questions

Launch

Students may need help focusing on the correct angles when there are multiple blocks involved. These students may benefit from using the Angle Window created in the warm-up for this lesson.

There are many ways to use the blocks to find the sizes of the angles in the first question. Students are encouraged to find more than one way, and to check that their answers remain the same.

Give students 2–3 minutes of quiet work time followed by a partner and whole-class discussion.

Representation: Internalise Comprehension. Begin with a physical demonstration of using pattern blocks to determine the size of an angle.
Supports accessibility for: Conceptual processing; Visual-spatial processing Writing, Conversing: Co-craft Questions. Use this routine to support language development through student conversations about mathematical questions. Without revealing the questions of the task, display only the image of the three angles for all to see. Invite students to work with a partner to write possible mathematical questions that could be asked about what they see. Listen for questions that connect the use of pattern blocks with measuring angles.
Design Principle(s): Cultivate conversation; Support sense-making

Anticipated Misconceptions

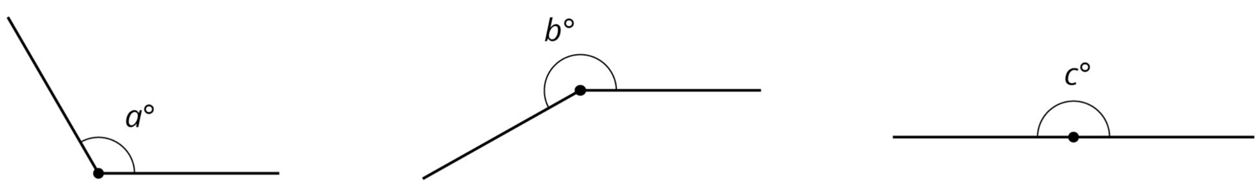
Some students may say that $b = 150$. Prompt them to notice that the arc marking which angle to measure is on the side that is greater than 180°.

If students are stuck on the angle that measures c degrees, consider using one of the patterns from the previous task that created a 360-degree angle with all the same pattern blocks and remove half of the pattern to show the 180-degree angle.

In the second problem, students might need encouragement to look for multiple combinations of pattern blocks to form a straight line.

Student Task Statement

1. Use pattern blocks to determine the size of each of these angles.



2. If an angle is 180°, then its sides form a straight line. An angle that forms a straight line is called a straight angle. Find as many different combinations of pattern blocks as you can that make a straight angle.

Student Response

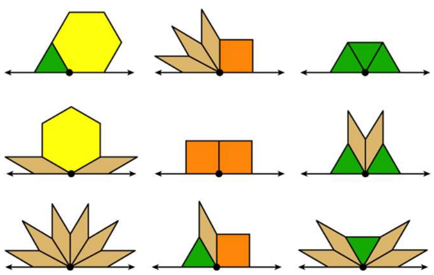
1. Explanations vary.

    a. 120° because it is the same size as one vertex of the yellow hexagon or two green triangles put together.

    b. 210° because it is the same size as one yellow hexagon and one orange square put together.

    c. 180° because it is the same size as three green triangles put together.

2. Answers vary. Sample responses:



Activity Synthesis

The goal of this discussion is for students to be exposed to many different examples of angles summing to 180°.

First, instruct students to compare their answers to the first question with a partner and share their reasoning until they reach an agreement. To help students see c as a 180-degree angle and not just a straight line, consider using only the smaller angle on the tan rhombus blocks to measure all three shapes: composing four tan rhombuses gives an angle measuring a degrees, seven rhombuses give an angle measuring b degrees, and six rhombuses give an angle measuring c degrees.

Next, select previously identified students to share their solutions to the second question. For each combination of blocks that is shared, invite other students in the class to write an equation displayed for all to see that reflects the reasoning.

1.4 Measuring Like This or That

Optional: 10 minutes

The purpose of this optional activity is to address the common error of reading a protractor from the wrong end. The problem gives students the opportunity to critique someone else's thinking and make an argument if they agree with either students' claim.

Instructional Routines

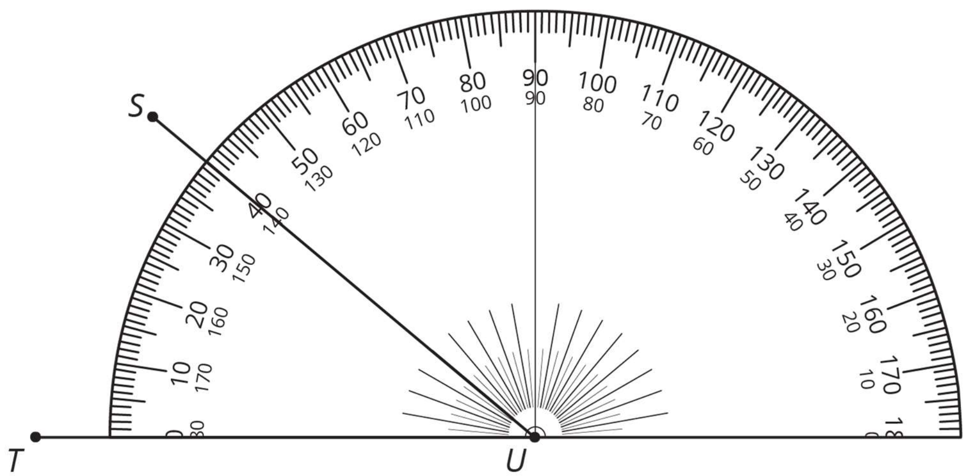
- Stronger and Clearer Each Time
- Think Pair Share

Launch

Arrange students in groups of 2. Give students 2–3 minutes of quiet think time followed by a partner and whole-class discussion.

Student Task Statement

Tyler and Priya were both measuring angle TUS.



Priya thinks the angle measures 40 degrees. Tyler thinks the angle measures 140 degrees. Do you agree with either of them? Explain your reasoning.

Student Response

Answers vary. Sample response: I agree with Priya, since the angle clearly measures less than 90 degrees. I think Tyler measured from the wrong end of the protractor.

Activity Synthesis

Ask students to indicate whether they agree with Priya or Tyler. Invite students to explain their reasoning until the class comes to an agreement that the measurement of angle TUS is 40 degrees.

Ask students how Tyler could know that his answer of 140 degrees is unreasonable for the size of angle TUS. Possible discussion points include:

- "Is angle TUS acute, right, or obtuse?" (acute)
- "Where is there an angle that measures 140 degrees in this shape?" (adjacent to angle TUS, from side US to the other side of the protractor)

Make sure that students understand that a protractor is often labelled with two sets of angle sizes, and they need to consider which side of the protractor they are measuring from.

Speaking, Listening, Conversing: Stronger and Clearer Each Time. Use this routine to give students a structured opportunity to revise and refine their explanation about whether or not they agree with Tyler or Priya. Give students time to meet with 2–3 partners, to share and get feedback on their responses. Provide students with prompts for feedback that will help their partners strengthen their ideas and clarify their language. For example, "What do you think each person did first?", "Could Priya and Tyler both be correct?", "Can you say that a different way?" Give students 1–2 minutes to revise their writing based on the feedback they received.
Design Principle(s): Cultivate conversation; Optimise output (for explanation)

Lesson Synthesis

- What are the three main types of angles in this lesson, and what are their sizes? (right: 90°, straight: 180°, all the way around a point: 360°)
- What does it look like when angles are adjacent, and what can you say about angle sizes? (The two angles are placed so that they share a vertex and one side. For adjacent angles, angles add. For example, a 60° angle adjacent to a 120° angle produces a 180° straight angle.)

1.5 Identical Isosceles Triangles

Cool Down: 5 minutes

Launch

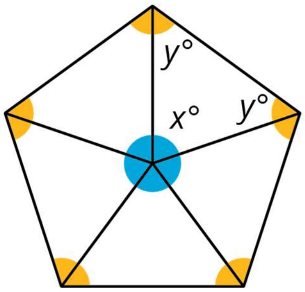
Consider displaying the image in colour to help students understand the image.

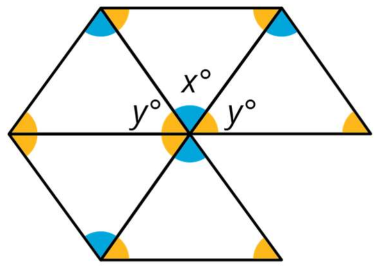
Anticipated Misconceptions

Some students may continue to struggle to understand the image, even after seeing the colour version. Help them mark all of the interior angles with either x or y. Alternatively, cut out the first shape and show how all the pieces can be rearranged to make the second shape.

Student Task Statement

Here are two different patterns made out of the same five identical isosceles triangles. Without using a protractor, determine the sizes of $\angle x$ and $\angle y$. Explain or show your reasoning.



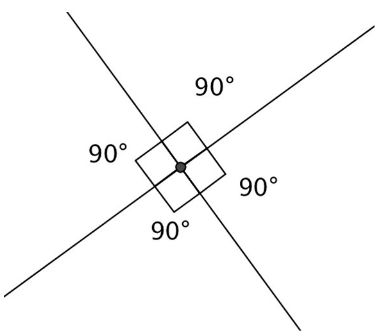


Student Response

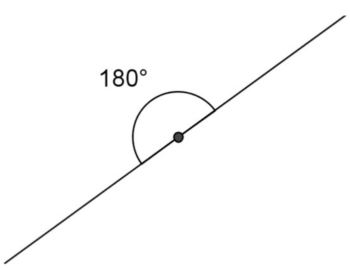
$x = 72$ and $y = 54$. Since there are 5 copies of the angle that measures x around a single point in the first picture, we know that $5x = 360$, so $x = 72$. In the second picture, we know that two copies of y and one copy of x make a straight angle, so $2y + 72 = 180$. Since we already know x, we can figure out that $y = 54$.

Student Lesson Summary

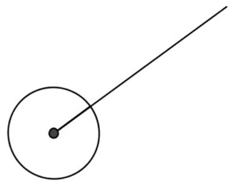
When two lines intersect and form four equal angles, we call each one a **right angle**. A right angle measures 90°. You can think of a right angle as a quarter turn in one direction or the other.



An angle in which the two sides form a straight line is called a **straight angle**. A straight angle measures 180°. A straight angle can be made by putting right angles together. You can think of a straight angle as a half turn, so that you are facing in the opposite direction after you are done.



If you put two straight angles together, you get an angle that is 360°. You can think of this angle as turning all the way around so that you are facing the same direction as when you started the turn.



When two angles share a side and a vertex, and they don't overlap, we call them **adjacent angles**.

Glossary

- adjacent angles
- right angle
- straight angle

Lesson 1 Practice Problems

1. **Problem 1 Statement**

   Here are questions about two types of angles.

   a. Draw a right angle. How do you know it's a right angle? What is its size in degrees?

   b. Draw a straight angle. How do you know it’s a straight angle? What is its size in degrees?

   Solution

   a. 90°. Responses vary. Sample responses: I used a protractor and measured; a square pattern block fits perfectly inside it; the corner of my notebook paper fits perfectly inside it.

   b. 180°. Responses vary. Sample response: I drew a straight line, and a straight angle is an angle formed by a straight line.

2. **Problem 2 Statement**

   An equilateral triangle’s angles each measures 60 degrees.

   a. Can you put copies of an equilateral triangle together to form a straight angle? Explain or show your reasoning.

   b. Can you put copies of an equilateral triangle together to form a right angle? Explain or show your reasoning.

   Solution

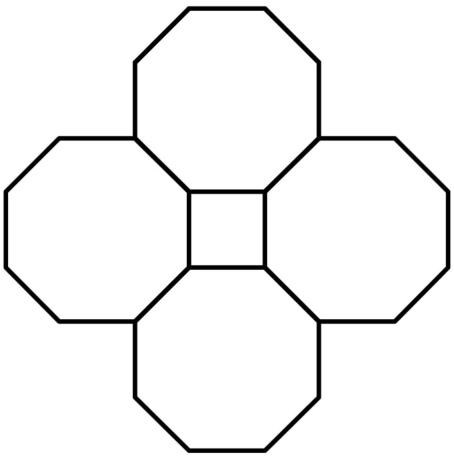
   a. Yes. 3 triangles are needed because $180 \div 3 = 60$.

b. No. One 60° angle is not enough, and two is too much.

3. **Problem 3 Statement**

Here is a square and some regular octagons.

In this pattern, all of the angles inside the octagons have the same size. The shape in the centre is a square. Find the size of one of the angles inside one of the octagons.



Solution

135°

4. **Problem 4 Statement**

The height of the water in a tank decreases by 3.5 cm each day. When the tank is full, the water is 10 m deep. The water tank needs to be refilled when the water height drops below 4 m.

a. Write a question that could be answered by solving the equation $10 - 0.035d = 4$.

b. Is 100 a solution of $10 - 0.035d > 4$? Write a question that solving this problem could answer.

Solution

Answers vary. Sample response:

a. "How many days can pass before the water tank needs to be refilled?"

b. Yes. "Is there still enough water in the tank after 100 days?"

5. **Problem 5 Statement**

Expand the brackets to write an expression that is equivalent to each given expression.

a. $\text{-}3(2x - 4)$

b. $0.1(\text{-}90 + 50a)$

c. $\text{-}7(\text{-}x - 9)$

d. $\frac{4}{5}(10y + \text{-}x + \text{-}15)$

Solution

a. $\text{-}6x + 12$

b. $\text{-}9 + 5a$

c. $7x + 63$

d. $8y - \frac{4}{5}x - 12$

6. **Problem 6 Statement**

Lin’s puppy is gaining weight at a rate of 0.125 pounds per day. Describe the weight gain in days per pound.

Solution

8 days per pound