Mathematics Education Innovation

Lesson 9: Gradients don't have to be positive

Goals

- Create a graph of a line representing a linear relationship with a non-positive rate of change.
- Interpret the gradient of a non-increasing line in context.

Learning Targets

- I can give an example of a situation that would have a negative gradient when graphed.
- I can look at a graph and tell if the gradient is positive or negative and explain how I know.

Lesson Narrative

In previous lessons, students have arrived at an equation for a line in three ways:

- By reasoning about similarity of gradient triangles on a line
- By reasoning about starting values and rates of change in a linear relationship
- By reasoning about vertical translations of lines through the origin

Students encountered linear relationships with positive rates of change and either positive or negative vertical intercepts. The graphs of these relationships all had an uphill appearance.

In this lesson, students get their first glimpse of lines that visually slope downhill as well as a “flat” line or line with 0 gradient. After reflecting on commonalities and differences between lines that gradient in different directions, students explore a situation in which one quantity decreases at a constant rate in relation to a second quantity. They interpret a graph of the situation and reason that it makes sense for the gradient to be negative in terms of the context. The scenario is then extended to consider a quantity that does not change with respect to another, and students realise that a flat graph has a gradient of zero.

Addressing

- Understand the connections between proportional relationships, lines, and linear equations.

Instructional Routines

- Stronger and Clearer Each Time
- Co-Craft Questions
- Three Reads
- Notice and Wonder

- Which One Doesn't Belong?

Required Materials

Geometry toolkit
tracing paper, graph paper, coloured pencils, scissors, and an index card to use as a straightedge or to mark right angles, a ruler and protractor. Clear protractors with no holes and with radial lines printed on them are recommended.

Student Learning Goals

Let's find out what a negative gradient means.

9.1 Which One Doesn't Belong: Odd Line Out

Warm Up: 5 minutes

This warm-up prompts students to compare four lines. It invites students to explain their reasoning and hold mathematical conversations, and allows you to hear how they use terminology and talk about lines and their properties. To allow all students to access the activity, each figure has one obvious reason it does not belong. Encourage students to move past the obvious reasons (e.g., line t has a different colour) and find reasons based on geometric properties (e.g., a gradient triangle of line u is not similar to the gradient triangles of the other three lines).

So far, we have only considered lines with positive gradients. The purpose of this warm-up is to suggest similarities (same vertical and horizontal lengths of gradient triangles) and differences (since they are not parallel, there is something fundamentally different going on here) between lines whose gradients have the same absolute value but opposite signs.

As students share their responses, listen for important ideas and terminology that will be helpful in the work of this lesson. Students may:

- Identify lines that have the same gradient.
- Identify points where lines intersect.
- Distinguish between lines with positive and negative values for gradient and be able to articulate the difference clearly.

Instructional Routines

- Which One Doesn't Belong?

Launch

Arrange students in groups of 2–4 and provide access to geometry toolkits. Display the image of the four lines for all to see. Ask students to indicate when they have noticed one line that does not belong and can explain why. Give students 2 minutes of quiet think time and then time to share their thinking with their group. After everyone has conferred in groups, ask each group to offer at least one reason a particular line doesn't belong.

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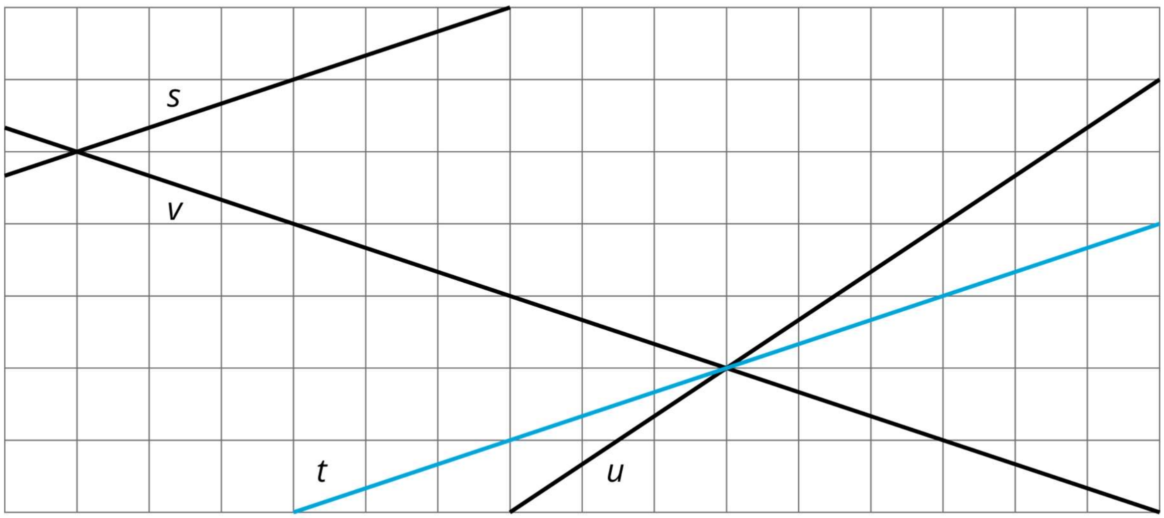
Student Task Statement

Which line doesn't belong?



Student Response

Answers vary. Possible responses:

- s doesn't belong because it doesn't go through that one point the rest of them do.
- t does not belong because it is parallel to s (and it is a different colour).
- u doesn't belong because its gradient triangle isn't similar to a triangle whose vertical side has length of 1 and whose horizontal side has length 3.
- v doesn't belong because it "leans to the left" instead of the right, or gradients down instead of up.

Activity Synthesis

After students have conferred in groups, invite each group to share one reason why a particular line might not belong. Record and display the responses for all to see. After each response, ask the rest of the class if they agree or disagree. Since there is no single correct answer to the question asking which shape does not belong, attend to students' explanations and ensure the reasons given are correct.

During the discussion, prompt students to use mathematical terminology (parallel, intersect, gradient) correctly. Also, press students on unsubstantiated claims. For example, a student may claim that u does not belong because it has a different gradient. Ask how they know for sure that its gradient is different from the other lines. Demonstrate drawing a gradient triangle and calculating gradient.

Based on the work done up to this point in the unit, students are likely to assume that the gradient of v is $\frac{1}{3}$. In the discussion, solicit the idea that there is something fundamentally different about line v compared to the others. You could use informal language like "uphill" and "downhill," or "tilt direction." The expressions positive and negative gradient do not need to be introduced at this time.

9.2 Stand Clear of the Closing Doors, Please

15 minutes (there is a digital version of this activity)

In previous activities with linear relationships, when x increases the y value increases as well; adding objects to a cylinder *increases* the water level and adding money to a bank account *increases* the balance. The gradient of the lines that represent these relationships were positive. In this activity, students see negative gradients for the first time.

In this activity, students answer questions about a public transport fare card context. After computing the amount left on the card after 0, 1, and 2 rides, they express regularity in repeated reasoning to represent the amount remaining on the card after x rides. They are told that the gradient of this line is -2.5, and are prompted to explain why a negative value makes sense.

While the language is not introduced in the task statement, the value of x for which the money on the card is 0 is called the x-intercept or horizontal intercept. Unlike the y-intercept, which can be seen in the equation $y = 40 - 2.5x$, the x intercept has to be calculated: it is the value of x for which $0 = 40 - 2.5x$.

Instructional Routines

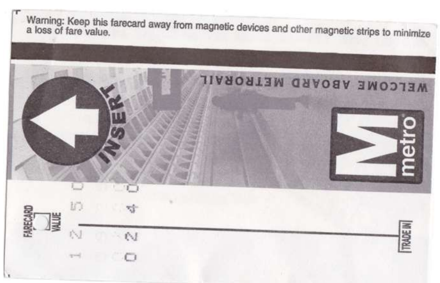
- Stronger and Clearer Each Time

Launch

If your students are unlikely to be familiar with public transport, you may need to give them some quick information about how a fare card works. If possible, prepare some photos related to purchasing and using a fare card. (Some example images are provided, here.)

Explain to students that someone who wants to ride the bus or train in a city often uses a card like this. The rider pays money which a computer system associates with the card. Every time the rider wants to ride, they swipe the card, and the cost of the ride is subtracted in the computer system from the balance on the card. Eventually, the amount available on the card runs out, and the rider must spend more money to increase the amount available on the card.

Arrange students in groups of 2 and give them 5 minutes of quiet work time. Provide access to rulers. After they have discussed their responses with a partner, discuss why a negative value for the gradient makes sense in the context and ways to visually tell whether a line has a positive or negative gradient.



Action and Expression: Internalise Executive Functions. Chunk this task into more manageable parts to support students who benefit from support with organisational skills in problem solving. For example, present one question at a time.
Supports accessibility for: Organisation; Attention

Student Task Statement

Noah put £40 on his fare card. Every time he rides on public transport, £2.50 is subtracted from the amount available on his card.

1. How much money, in pounds, is available on his card after he takes
   a. 0 rides?
   b. 1 ride?
   c. 2 rides?
   d. x rides?
2. Graph the relationship between amount of money on the card and number of rides.

3. How many rides can Noah take before the card runs out of money? Where do you see this number of rides on your graph?

Student Response

1. Money stored on card in pounds:

   a. 40

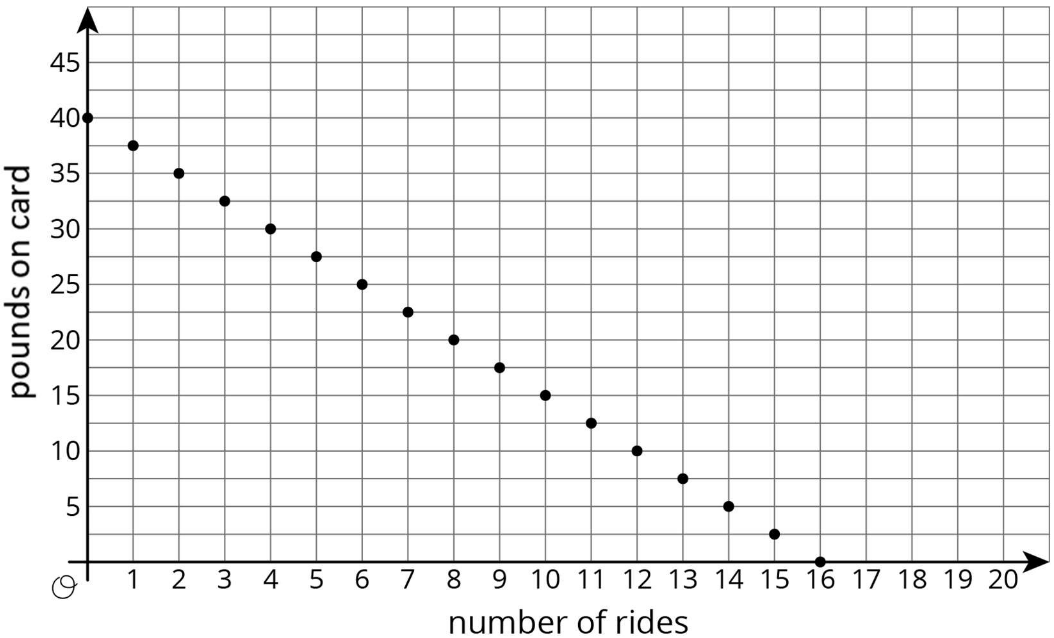
   b. 37.5

   c. 35

   d. $40 - 2.5x$ (or equivalent)

2. See graph. A graph that shows only points with integer values of number of rides is also acceptable.

3. 16 rides. The point (16,0) is on the graph.



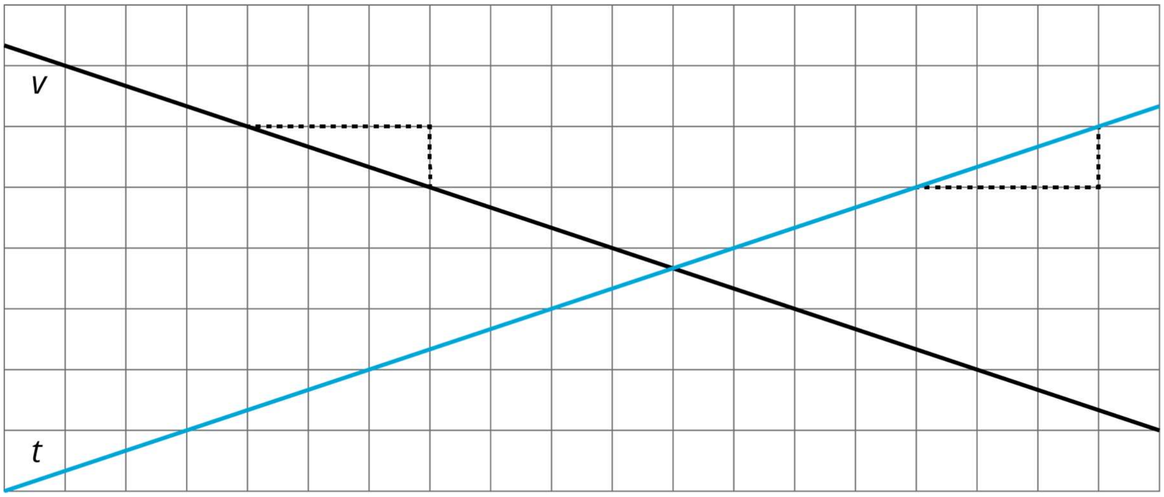
Activity Synthesis

Ask students, "Why does it make sense to say the gradient of this graph is -2.5 rather than 2.5?" We can say that the rate of change of the amount on the card is -2.5 pounds per ride (for each ride, the amount on the card *decreases* by 2.5). We can also say the gradient of the graph representing the relationship is -2.5 (when x increases by 1, y decreases by 2.5). Ensure that students can articulate some version of "For every ride, the amount on the card *decreased* by £2.50, and this is why it makes sense that the gradient is -2.5."

Ask students, "If we let y represent the amount of money on the card, in pounds, and x the number of rides, the linear relationship in this activity has the equation $y = 40 - 2.5x$. When will there be no money left on the card?" (After 16 rides, the card has no money left, so when $x = 16$, $y = 0$.) Explain that just like the point (0,40) on the graph is called the vertical intercept, the point (16,0) is called the horizontal intercept. In this situation, the vertical intercept tells us how much money Noah put on the card and the horizontal intercept tells us how many rides Noah can take before the card has no more money on it.

Some students may connect the plotted points with a line, or only draw the line and not each point. If this comes up, acknowledge that it's a common practice in mathematical modelling to connect points showing a line to make a relationship easier to see. Ask if a point on the line such as (1.5,36.25) makes sense in this situation. We understand that the line helps us to understand the situation, but we also recognise that only integer values for the number of rides make sense in this situation.

Display once again lines v and t from the warm-up. Explain that even though they both have gradient triangles similar to a triangle with vertical length of 1 and horizontal length of 3, they don't have the same gradient. Ask students to articulate which line has a gradient of $\frac{1}{3}$ and which line has a gradient of $\frac{-1}{3}$, and why. Validate students' use of informal language to describe the differences in their own words. For example, they might say that if you put a pencil on a line and move it along the line from left to right, line t goes "uphill" but line v goes "downhill."



Writing, Conversing: Stronger and Clearer Each Time. Use this routine to support students to explain their reasoning for the question, "Why does it make sense to say the gradient of this graph is -2.5 rather than 2.5?" Give students time to meet with 2–3 partners, to share and get feedback on their responses. Encourage listeners to ask their partner clarifying questions such as, "Can you demonstrate your thinking by using the graph?" and "How does this negative value make sense for this situation?" Allow students to write a revised draft that reflects ideas and language from their shares. This will help students refine their thinking through conversing with their partners.
Design Principle(s): Optimise output (for explanation); Cultivate conversation

9.3 Travel Habits in July

10 minutes (there is a digital version of this activity)

The previous activity examines the meaning of a negative gradient with the context of money on a fare card. This activity takes advantage of familiarity with the same context, introducing the idea of 0 gradient. In the previous activity, the gradient of -2.5 meant that for every ride, the amount on the card decreased by £2.50. In this activity, the amount on the card is graphed with respect to days in July. For every new day, the amount on the card does not change; it does not go up or down at all. The purpose is for students to understand the meaning of a gradient of 0 in this context.

Instructional Routines

- Co-Craft Questions
- Notice and Wonder

Launch

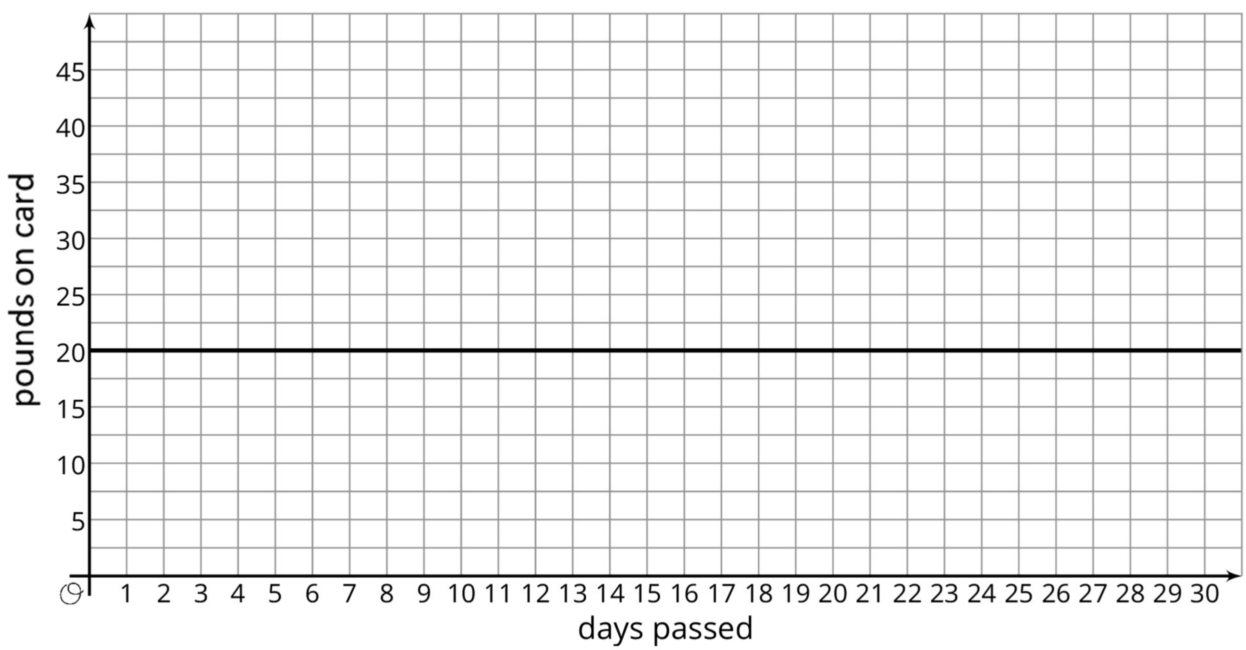
Show the image and ask students "What do you notice? What do you wonder?" Expect students to notice that the line is horizontal (or the amount of money on the card does not change). They may wonder why the line is horizontal or what its gradient is.

Keep students in the same groups. 5 minutes of quiet think time, followed by partner and class discussion. Before students begin the task, ensure that they understand that the x-axis no longer represents number of rides, but rather, different days in July.

Action and Expression: Internalise Executive Functions. Provide students with a graphic organiser to record what they notice and wonder prior to being expected to share these ideas with others.
Supports accessibility for: Language; Organisation Writing, Speaking: Co-Craft Questions.
Present the graph that shows the amount on Han's fare card for every day of July, without revealing the questions that follow. Give students time to write down possible mathematical questions that can be asked about the situation, and then invite them to share their questions with a partner. Listen for and amplify questions that wonder about the gradient of a horizontal line. This will help students make sense of a graph with a "flat line," or a line having a zero gradient, by generating questions using mathematical language.
Design Principle(s): Cultivate conversation; Maximise meta-awareness

Student Task Statement

Here is a graph that shows the amount on Han's fare card for every day of July.



1. Describe what happened with the amount on Han's fare card in July.

2. Plot and label 3 different points on the line.

3. Write an equation that represents the amount on the card in July, y, after x days.

4. What value makes sense for the gradient of the line that represents the amounts on Han's fare card in July?

Student Response

1. The amount on the card was £20 every day. The amount on the card did not change.

2. Answers vary. Possible points: (0,20), (2,20), and (30,20).

3. $y = 20$

4. The gradient of the graph is 0. The rate of change is £0 per day. Also, the line is neither "uphill" nor "downhill," so it makes sense that the gradient is neither positive nor negative.

Are You Ready for More?

Let's say you have taken out a loan and are paying it back. Which of the following graphs have positive gradient and which have negative gradient?

1. Amount paid on the vertical axis and time since payments started on the horizontal axis.

2. Amount owed on the vertical axis and time remaining until the loan is paid off on the horizontal axis.

3. Amount paid on the vertical axis and time remaining until the loan is paid off on the horizontal axis.

Student Response

1. Positive gradient. At the beginning, the amount paid is nothing and the time that has passed is zero. Later, the amount paid is more and the time that has passed is more, so the graph gradients upward.

2. Positive gradient. If the time remaining is zero, then you are at the end of the loan, and so the amount owed is also zero. So you are at the origin on the graph. If time remaining is greater than zero then something is owed, so you are at a point with both coordinates positive. So the graph has to gradient upward.

3. Negative gradient. If the time remaining is zero, then you are at the end of the loan, and the amount paid is the whole loan. So you are at a point on the positive vertical axis. If the time remaining is the entire period of the loan, then you are at the beginning of the loan, and the amount paid is zero. So you are at a point on the positive horizontal axis. So the graph has to gradient down.

Activity Synthesis

Display the graph for all to see, and ask students to articulate what they think the gradient of the graph is, and why. The goal is to understand that a gradient of 0 makes sense because no money is added or subtracted each day. If students have been thinking in terms of uphill and downhill lines, they might describe this line as "flat," indicating that the gradient can't be positive or negative, so 0 makes sense. Thinking of gradient as the quotient of horizontal displacement by vertical displacement for two points on a line is very effective here: the vertical displacement is 0 for any two points on this line, and so the quotient or gradient is also 0.

If no one brings it up, ask students what would happen if we tried to create a gradient triangle for this line. They might claim that it would be impossible, but suggest that we can think of a gradient "triangle" where the vertical segment has length 0. In other words, it is possible to imagine a horizontal line segment as a triangle whose base is that segment and whose height is 0. If possible, display and demonstrate with the following: https://ggbm.at/vvQPutaJ.

9.4 Payback Plan

Optional: 10 minutes

This activity gives students an opportunity to interpret the graph of a line in context, including the meaning of a negative gradient and the meaning of the horizontal and vertical intercepts.

Instructional Routines

- Three Reads

Launch

3 minutes of quiet work time, followed by 2 minutes to confer in small groups to verify answers.

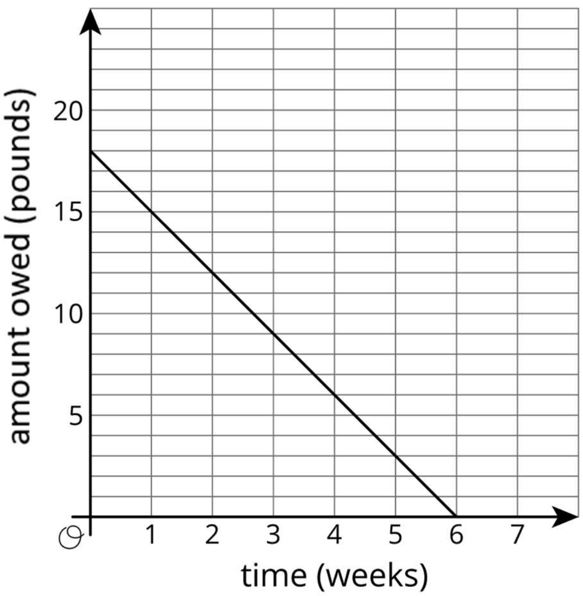
Reading: Three Reads. Use this routine for Elena's situation to support reading comprehension for students. In the first read, students read the problem with the goal of comprehending the situation (e.g., Elena is paying back her brother every week.). In the second read, ask students to look for quantities represented in the graph (e.g., at week 0, Elena owes him £18; at 6 weeks, she has paid him back completely). In the third read, ask students to brainstorm possible strategies to answer the question: "What is the gradient of the line and what does it represent?" This will help students reflect on the situation and interpret the gradient in the given context.
Design Principle(s): Support sense-making; Maximise meta-awareness

Anticipated Misconceptions

Students may interpret money owed as negative. In this setup, the axis is labelled so that money owed is treated as positive.

Student Task Statement

Elena borrowed some money from her brother. She pays him back by giving him the same amount every week. The graph shows how much she owes after each week.



Answer and explain your reasoning for each question.

1. What is the gradient of the line?
2. Explain how you know whether the gradient is positive or negative.
3. What does the gradient represent in this situation?
4. How much did Elena borrow?
5. How much time will it take for Elena to pay back all the money she borrowed?

Student Response

1. The gradient is -3.
2. Answers vary. Sample response: Negative, because the line gradients down. Negative, because the amount owed is decreasing as time increases.
3. Answers vary. Sample response: The gradient represents the amount that she pays back each week. The gradient represents how much less she owes every time a week goes by.
4. She borrowed £18, which we see from the point (0,18) on the graph.
5. It took her 6 weeks to pay back the £18, which we can see from the point (6,0) on the graph.

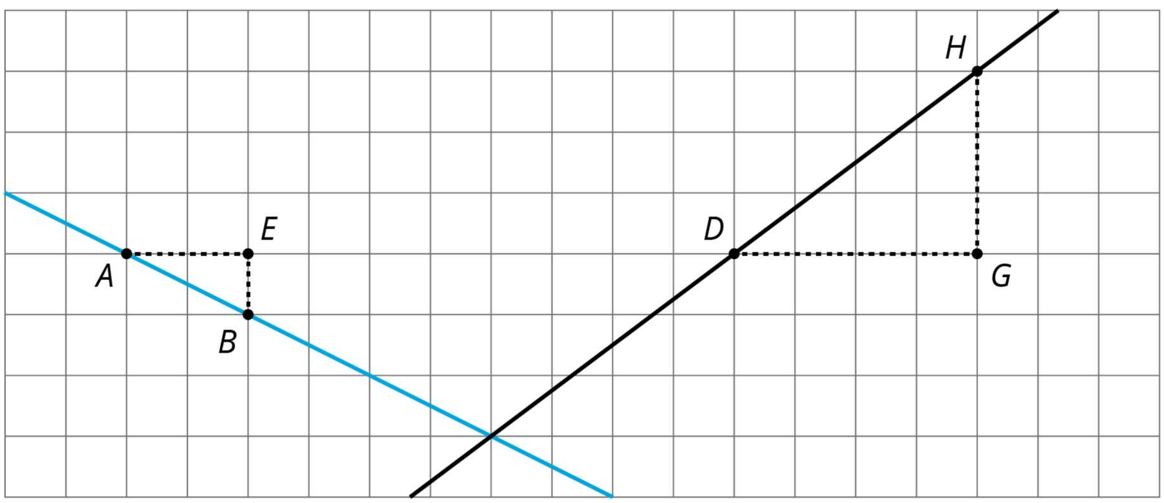
Activity Synthesis

Ask students to share answers to each question and indicate how to use the graph to find the answers. For example, draw a gradient triangle for the first question: the gradient is -3 rather than 3, because the number of pounds owed is decreasing over time. Label the vertical intercept for the amount Elena borrowed and the horizontal intercept for the time it took her to pay back the loan.

Lesson Synthesis

In this lesson, students learned that the gradient of a line can be a negative value or 0. They saw some linear relationships with a negative gradient and some with 0 gradient. Students learned about cues to identify whether a graphed line has a positive gradient, a negative gradient, or 0 gradient.

Display the graph for all to see. Ask students to pretend that their partner has been absent from class for a few days. Their job is to explain, verbally or in writing, how someone would figure out the gradient of one of the graphed lines. Then, switch roles and listen to their partner explain how to figure out the gradient of the other line.



9.5 The Gradients of Graphs

Cool Down: 5 minutes

Student Task Statement

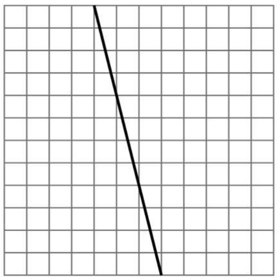
Each square on a grid represents 1 unit on each side.

1. Calculate the gradient of graph D. Explain or show your reasoning.

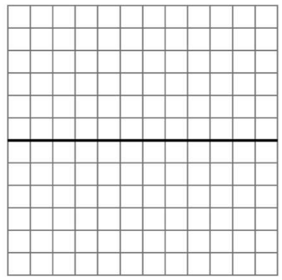
2. Calculate the gradient of graph E. What situation could the graph represent?

3. On the blank grid F, draw a line that passes through the indicated point and has gradient -2.

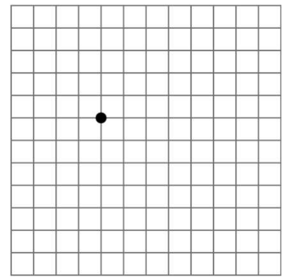
D



E



F



Student Response

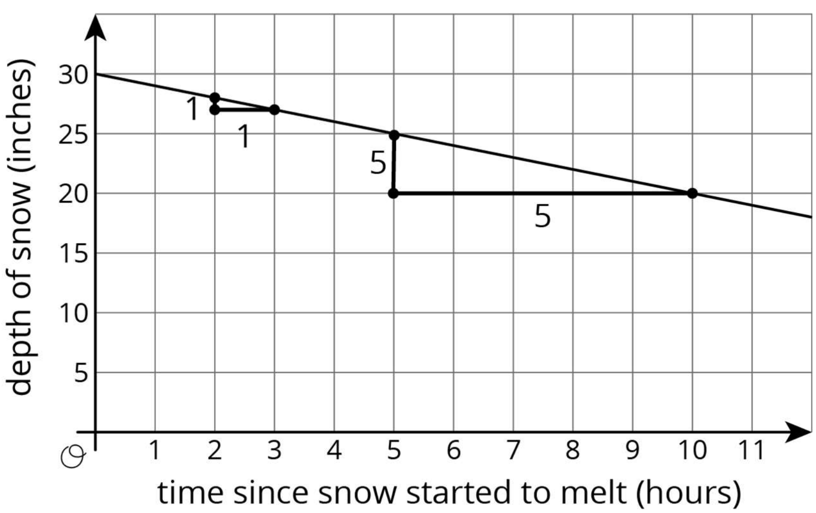
1. -4. Explanations vary. Sample explanation: for each horizontal change of 1, the vertical change is -4, so the gradient of the line is -4.

2. 0. Responses vary. Sample response: The amount of rainfall on a day with no rain.

3. A graph through the indicated point with a gradient of -2.

Student Lesson Summary

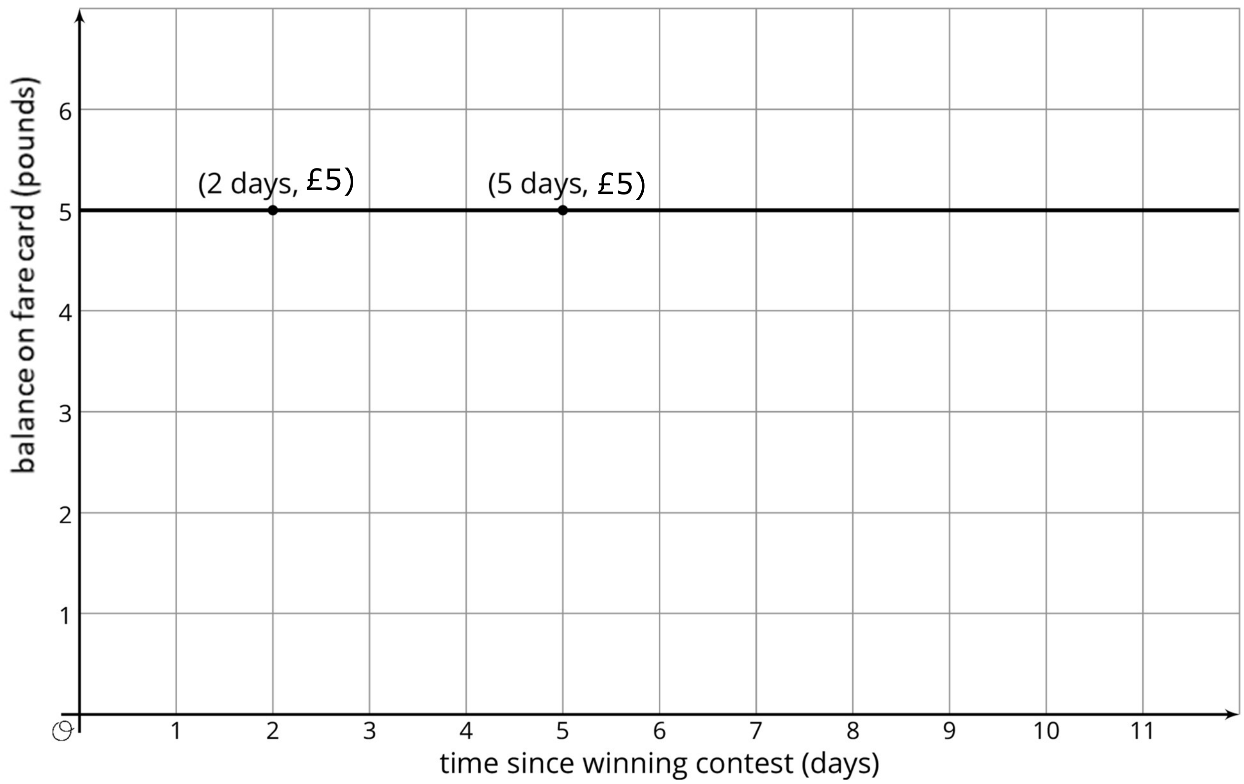
At the end of winter in Maine, the snow on the ground was 30 inches deep. Then there was a particularly warm day and the snow melted at the rate of 1 inch per hour. The graph shows the relationship between the time since the snow started to melt and the depth of the snow.



The gradient of the graph is -1 since the rate of change is -1 inch per hour. That is, the depth goes *down* 1 inch per hour. The vertical intercept is 30 since the snow was 30 inches deep when the warmth started to melt the snow. The two gradient triangles show how the rate of change is constant. It just also happens to be negative in this case since after each hour that passes, there is 1 inch *less* snow.

Graphs with negative gradient often describe situations where some quantity is decreasing over time, like the depth of snow on warm days or the amount of money on a fare card being used to take rides on buses.

Gradients can be positive, negative, or even zero! A gradient of 0 means there is no change in the y-value even though the x-value may be changing. For example, Elena won a contest where the prize was a special pass that gives her free bus rides for a year. Her fare card had £5 on it when she won the prize. Here is a graph of the amount of money on her fare card after winning the prize:



The vertical intercept is 5, since the graph starts when she has £5 on her fare card. The gradient of the graph is 0 since she doesn't use her fare card for the next year, meaning the amount on her fare card doesn't change for a year. In fact, all graphs of linear relationships with gradients equal to 0 are horizontal—a rate of change of 0 means that, from one point to the next, the y-values remain the same.

Lesson 9 Practice Problems

1. **Problem 1 Statement**

   Suppose that during its flight, the height e (in feet) of a certain airplane and its time t, in minutes since takeoff, are related by a linear equation. Consider the graph of this equation, with time represented on the horizontal axis and height on the vertical axis. For each situation, decide if the gradient is positive, zero, or negative.

   a. The plane is cruising at an altitude of 37 000 feet above sea level.

b. The plane is descending at rate of 1 000 feet per minute.

c. The plane is ascending at a rate of 2 000 feet per minute.

Solution

a. Zero

b. Negative

c. Positive

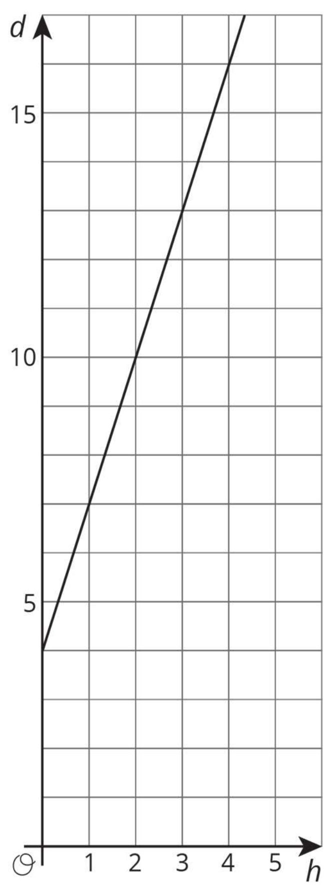
2. **Problem 2 Statement**

A group of hikers park their car at a trail head and walk into the forest to a campsite. The next morning, they head out on a hike from their campsite walking at a steady rate. The graph shows their distance in miles, d, from the car after h hours of hiking.

a. How far is the campsite from their car? Explain how you know.

b. Write an equation that describes the relationship between d and h.

c. After how many hours of hiking will they be 16 miles from their car? Explain or show your reasoning.



Solution

a. 4 miles. The y-intercept represents this initial distance before the start of the hike.

b. $d = 4 + 3h$

c. 4 hours. Explanations vary. Sample response: On the graph, $d = 16$ when $h = 4$. The equation can be used to solve for h when $d = 16$
$16 = 4 + 3h$, $12 = 3h$ and $h = 4$.

3. **Problem 3 Statement**

Elena's aunt pays her £1 for each call she makes to let people know about her aunt's new business.

The table shows how much money Diego receives for washing windows for his neighbours.

number of windows	number of pounds
27	30
45	50
81	90

Select **all** the statements about the situation that are true.

a. Elena makes more money for making 10 calls than Diego makes for washing 10 windows.

b. Diego makes more money for washing each window than Elena makes for making each call.

c. Elena makes the same amount of money for 20 calls as Diego makes for 18 windows.

d. Diego needs to wash 35 windows to make as much money as Elena makes for 40 calls.

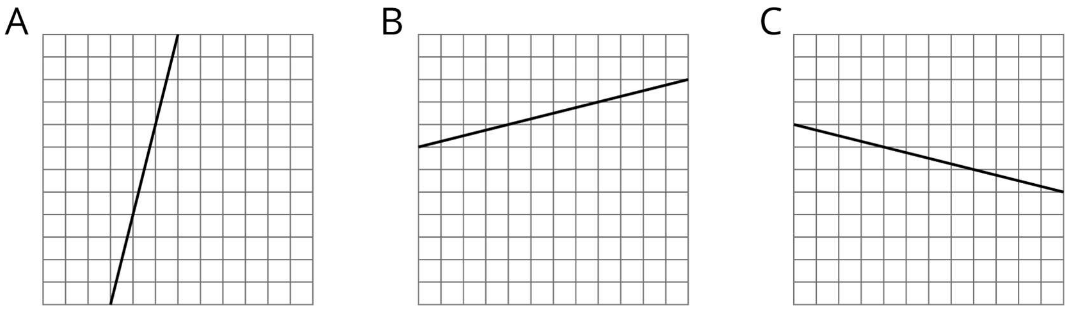
e. The equation $y = \frac{9}{10}x$, where y is number of pounds and x is number of windows, represents Diego's situation.

f. The equation $y = x$, where y is the number of pounds and x is the number of calls, represents Elena's situation.

Solution ["B", "C", "F"]

4. **Problem 4 Statement**

Each square on a grid represents 1 unit on each side. Match the graphs with the gradients of the lines.



- $-\frac{1}{4}$
- $\frac{1}{4}$
- 4

Solution

A has gradient 4, B has gradient $\frac{1}{4}$, C has gradient $-\frac{1}{4}$