Lesson 1: Projecting and scaling

Goals

- Comprehend the term "enlargement" as a process that produces scaled copies.
- Describe (orally) features of scaled copies of a rectangle.
- Identify rectangles that are scaled copies of one another.

Learning Targets

- I can decide if one rectangle is an enlargement of another rectangle.
- I know how to use a centre and a scale factor to describe an enlargement.

Lesson Narrative

Previously students examined scaled copies. For polygons, they identify that side lengths of scaled copies are proportional, and the constant of proportionality relating the original lengths to the corresponding lengths in the scaled copy is the scale factor. This lesson builds on this experience. In the first activity, students arrange a set of scaled copies of rectangles and observe that if the rectangles are arranged to share one angle, then the opposite vertices all lie on the same line. This motivates an informal introduction of *enlargement*, a geometric process that produces scaled copies. In the context of the set of rectangles, the shared vertex is the *centre of enlargement* and, as students will learn in later lessons, the enlargement scales the distance of all points (not just the upper right vertex of the rectangle) from the centre of enlargement. A second optional activity recalls explicitly earlier work from about scaled copies of rectangles.

Building On

- Apply and extend previous understandings of multiplication and division to divide fractions by fractions.
- Solve problems involving scale drawings of geometric shapes, including computing actual lengths and areas from a scale drawing and reproducing a scale drawing at a different scale.

Addressing

- Understand congruence and similarity using physical models, transparencies, or geometry software.

Building Towards

- Understand congruence and similarity using physical models, transparencies, or geometry software.

Instructional Routines

- Collect and Display

- Co-Craft Questions
- Compare and Connect
- Discussion Supports
- Number Talk

Required Materials

Blank paper
Four-function calculators
Long straightedge
Rulers marked with inches
Scissors

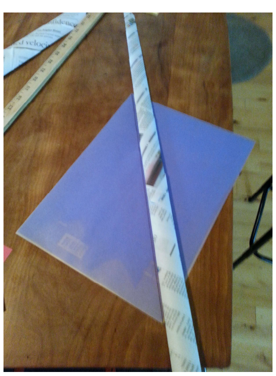
Required Preparation

For the activity Sorting Rectangles, decide whether students will create their own set of rectangles A–E or if you will create these ahead of time. If students will create their own, they need 2 sheets of copier paper and a pair of scissors. (Students do not need scissors if they are not creating the rectangles.) If you will create them ahead of time, prepare and label one set A–E for each pair of students:

- A: One full sheet, 8.5 by 11 inch
- B: One half sheet, 8.5 by 5.5
- C: One quarter sheet, 4.25 by 5.5
- D: One eighth sheet, 4.25 by 2.75
- E: One sixteenth sheet, 2.125 by 2.75

Calculators are optional. Decide whether you want students to handle the calculations without a calculator or whether you will offer calculators.

Each pair of students will also need a long straightedge (at least 14 inches long). Metre rules or yardsticks would work, or a long straightedge can be created from newspaper, like this:



Student Learning Goals

Let's explore scaling.

1.1 Number Talk: Remembering Fraction Division

Warm Up: 10 minutes

This Number Talk gives students an opportunity to recall strategies for calculation problems that will arise in the lesson. While many strategies may emerge, the focus of these problems is for students to recall and rehearse a reliable way to divide a mixed number by a whole number. Likely strategies are:

- Use the brackets to divide each component of the mixed number separately.
- Convert the mixed number into a fraction of the form $\frac{a}{b}$, then multiply by the reciprocal of the divisor.

Three problems are given. In the limited time available, however, it may not be possible to share every possible strategy. Consider gathering only one or two different strategies per problem.

Instructional Routines

- Collect and Display
- Compare and Connect
- Discussion Supports
- Number Talk

Launch

Display one problem at a time. Give students 1 minute of quiet think time per problem and ask them to give a signal when they have an answer and a strategy. Follow with a whole-class discussion.

Representation: Internalise Comprehension. To support working memory, provide students with sticky notes or mini whiteboards.
Supports accessibility for: Memory; Organisation

Student Task Statement

Find each quotient. Write your answer as a fraction or a mixed number.

$6\frac{1}{4} \div 2$

$10\frac{1}{7} \div 5$

$8\frac{1}{2} \div 11$

Student Response

- $3\frac{1}{8}$ or $\frac{25}{8}$. Possible strategies:

    a. $6 \div 2 = 3$ and $\frac{1}{4} \div 2 = \frac{1}{8}$

    b. $6\frac{1}{4} = \frac{25}{4}$, and $\frac{25}{4} \div 2 = \frac{25}{4} \times \frac{1}{2} = \frac{25}{8}$

- $2\frac{1}{35}$ or $\frac{71}{35}$. Possible strategies:

    a. $10 \div 5 = 2$ and $\frac{1}{7} \div 5 = \frac{1}{35}$

    b. $10\frac{1}{7} = \frac{71}{7}$, and $\frac{71}{7} \div 5 = \frac{71}{7} \times \frac{1}{5} = \frac{71}{35}$

- $\frac{17}{22}$. Possible strategy:

    a. $8 \div 11 = \frac{8}{11}$, $\frac{1}{2} \div 11 = \frac{1}{22}$, and $\frac{8}{11} + \frac{1}{22} = \frac{17}{22}$

    b. $8\frac{1}{2} = \frac{17}{2}$, and $\frac{17}{2} \div 11 = \frac{17}{2} \times \frac{1}{11} = \frac{17}{22}$

Activity Synthesis

Ask students to share their strategies for each problem. Record and display their explanations for all to see. Ask students if or how the numbers in the problem impacted their choice of strategy. Point out how the first problem differs from the third in an important way: for the first problem it is not necessary to convert the mixed number to a fraction because the whole number part of the mixed number is evenly divisible by 2.

To involve more students in the conversation, use *Compare and Connect* by asking probing questions and connecting students' responses, such as:

- "Who can restate ____'s reasoning using your own words?"
- "Does anyone want to add on to ____'s explanation?"
- "Do you agree or disagree ____'s reasoning? Why?"
- "How is ____'s reasoning similar to and different from ____'s reasoning?"

Speaking: Discussion Supports.: Display sentence frames to support students when they explain their strategy. For example, "First, I ____ because . . ." or "I noticed ____ so I . . . ." Some students may benefit from the opportunity to rehearse what they will say with a partner before they share with the whole class.
Design Principle(s): Optimise output (for explanation)

1.2 Sorting Rectangles

20 minutes

This activity recalls earlier work on scaled copies, purposefully arranging a set of scaled copies to prepare students to understand the process of enlargement. If one rectangle is a scaled copy of another, then they can be arranged so that the diagonal of the larger rectangle *contains* the diagonal of the smaller rectangle. To do this, it is sufficient to line up the rectangles so that the vertices of the right angles at their lower left match up. Students will arrange a set of rectangles into groups with shared diagonals and examine the scale factors relating the rectangles. Afterward, during the discussion, the word enlargement is first used, in an informal way, as a way to make scaled copies (of the rectangle in this case). From this point of view, the shared vertex of each set of rectangles is the centre of enlargement and once we choose an original rectangle from each set, there is a scale factor associated to each copy, namely the scale factor needed to produce the copy from the original.

As an *optional* additional part to this activity, students may perform a visual test that helps decide whether or not two cut-out shapes are scaled copies of one another. The visual test tells whether two cut-out shapes are scaled copies of each other by holding each shape at a different distance from the eye and checking if it is possible to make the two shapes match up exactly.

Monitor for how students sort the rectangles and how they find measurements of the new rectangles. Encourage them to use what they know about how the rectangles were created rather than measuring each new rectangle (which is likely to introduce errors). Also monitor for how they decide if one rectangle is a scaled copy of another.

Instructional Routines

- Collect and Display

Launch

If students will perform the optional eyeball test, tell them that one way to check whether two shapes are scaled copies of each other is to use the “eyeball test.” Students will perform this test for themselves in the activity, but will watch a demonstration first.

- Hold one rectangle up in front of your face in one hand, and another rectangle farther away from your face in the other hand. (The larger rectangle should be farther away than the smaller rectangle.)
- Close one eye.
- If you can adjust your arms so that the rectangles appear to be exactly the same, then they are a match. If it is not possible to adjust your arms so that the rectangles appear to be exactly the same, then they are not a match. Explain to students that their job will be to use the eyeball test to shape out which pairs of their rectangles are matches.

Arrange students in groups of 2. Provide a set of 5 pre-cut rectangles and a long straightedge to each group and, optionally, access to calculators. (Alternatively, you could give each group two whole sheets of paper and instruct them to do the folding and cutting. This option might be attractive if your students would understand the idea of "halving" the measurements better with the concrete experience.)

If students are performing the optional "eyeball test" on pairs of rectangles, instruct them to sort the rectangles into different piles so that all of the rectangles in each pile "match" one another according to the eyeball test. Instruct them to discuss their thinking as a group to reach an agreement.

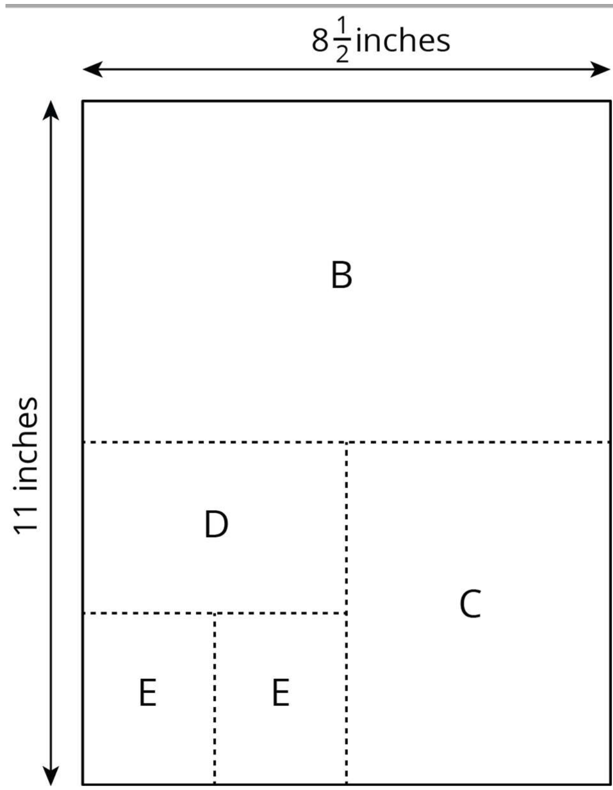
Representation: Internalise Comprehension. Provide a range of examples and counterexamples of rectangles that are scaled copies of the full sheet of paper. Show an example of a scaled copy and then a counterexample. Be sure to justify the reasoning in each example. Consider providing step-by-step directions for students to find the scale factor between rectangles and how to compare corresponding angle measures.
Supports accessibility for: Conceptual processing Conversing, Reading: Collect and Display. As students work in pairs to make sense of the problem, circulate and listen to students as they determine which rectangles are scaled copies of the full sheet of paper. Write down the observations students make about the measurements of rectangles A, C, and E. As students review the language and diagrams collected in the visual display, encourage students to clarify the meaning of a word or phrase. For example, a phrase such as "rectangle E is the same as rectangle C but smaller" can be clarified by rephrasing the statement as "the side lengths of rectangle E are half of the side lengths of rectangle C." This routine will provide feedback to students in a way that supports sense-making while simultaneously increasing meta-awareness of language.
Design Principle(s): Support sense-making; Maximise meta-awareness

Anticipated Misconceptions

If students forget how to check if two rectangles are scaled copies of one another, remind them to compare the measurements to see if they have the same scale factor. Students may recall that that scaled copies have corresponding angles of the same size, but they may not recall that equal angle measurements don't necessarily mean you have scaled copies.

Student Task Statement

Rectangles were made by cutting an $8\frac{1}{2}$-inch by 11-inch piece of paper in half, in half again, and so on, as illustrated in the diagram. Find the lengths of each rectangle and enter them in the appropriate table.



1. Some of the rectangles are scaled copies of the full sheet of paper (rectangle A). Record the measurements of those rectangles in this table.

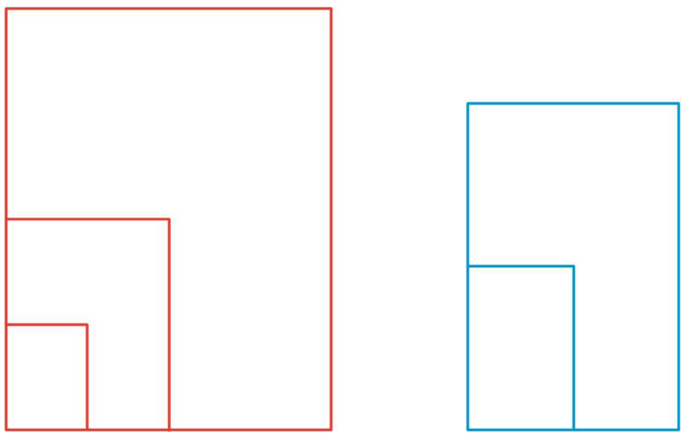
rectangle	length of short side (inches)	length of long side (inches)
A	$8\frac{1}{2}$	11

2. Some of the rectangles are *not* scaled copies of the full sheet of paper. Record the measurements of those rectangles in this table.

rectangle	length of short side (inches)	length of long side (inches)

3. Look at the measurements for the rectangles that are scaled copies of the full sheet of paper. What do you notice about the measurements of these rectangles? Look at the measurements for the rectangles that are *not* scaled copies of the full sheet. What do you notice about these measurements?

4. Stack the rectangles that are scaled copies of the full sheet so that they all line up at a corner, as shown in the diagram. Do the same with the other set of rectangles. On each stack, draw a line from the bottom left corner to the top right corner of the biggest rectangle. What do you notice?



5. Stack *all* of the rectangles from largest to smallest so that they all line up at a corner. Compare the lines that you drew. Can you tell, from the drawn lines, which set each rectangle came from?

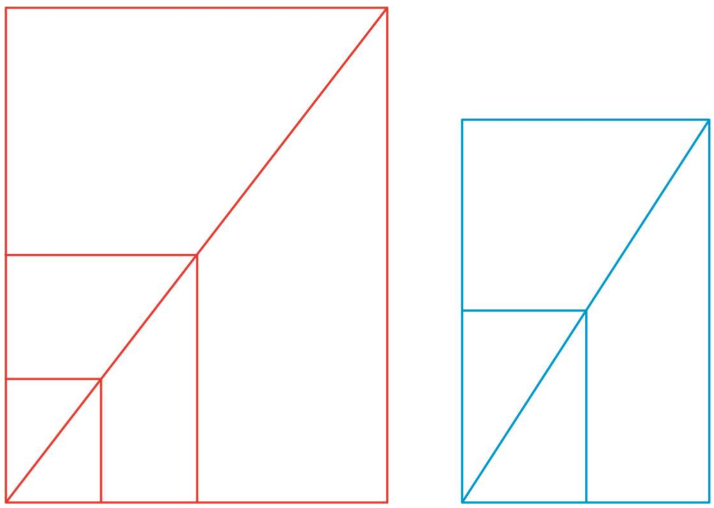
Student Response

If students perform the eyeball test, then rectangles A, C, and E are matches. B and D also match.

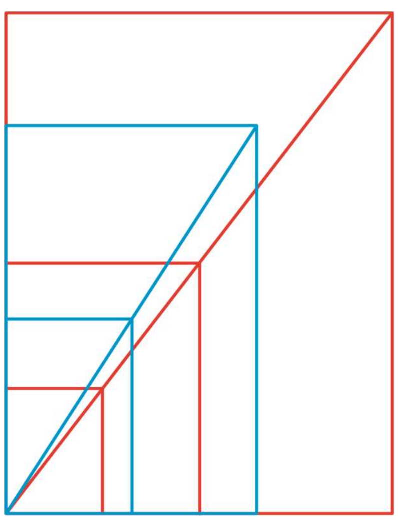
rectangle	length of short side (inches)	length of long side (inches)
A	$8\frac{1}{2}$	11
C	$4\frac{1}{4}$	$5\frac{1}{2}$
E	$2\frac{1}{8}$	$2\frac{3}{4}$
rectangle	length of short side (inches)	length of long side (inches)
B	$5\frac{1}{2}$	$8\frac{1}{2}$
D	$2\frac{3}{4}$	$4\frac{1}{4}$

3. The lengths of the short and long sides of each set of rectangles form a set of equivalent ratios. There are different ways of expressing this idea: students might reason from column to column and say that the measurements have the same quotient, the same unit rate, or the same constant of proportionality. For rectangles A, C, and E, the quotient of the lengths of a short side and a long side is $\frac{17}{22}$. For rectangles B and D, this quotient is $\frac{11}{17}$. They also might reason multiplicatively from row to row. For example, from A to C, you can multiply each measurement in A by $\frac{1}{2}$ to get the corresponding measurement in C.

4. The diagonal goes through two vertices of each rectangle in the pile.



5. Yes; the diagonals of the second set lie above the diagonals of the first set.



Are You Ready for More?

In many countries, the standard paper size is not 8.5 inches by 11 inches (called "letter" size), but instead 210 millimetres by 297 millimetres (called "A4" size). Are these two rectangle sizes scaled copies of one another?

Student Response

No. Converting from millimetres to inches, A4 paper is about 8.27 inches by 11.69 inches. Since it is both taller and less wide than letter paper, it could not be a scaled copy.

Activity Synthesis

Ask students how they decided that the $5\frac{1}{2}$ by $8\frac{1}{2}$ rectangle is not a scaled copy of the $8\frac{1}{2}$ by 11 rectangle. Make sure to provide a mathematical explanation since it is not easy to determine visually. For example, there is no single number that you can multiply by $5\frac{1}{2}$ to get $8\frac{1}{2}$ *and* multiply by $8\frac{1}{2}$ to get 11.

Ask students how they decided that the $5\frac{1}{2}$ by $4\frac{1}{4}$ rectangle is a scaled copy of the 11 by $8\frac{1}{2}$ rectangle, and again, emphasise mathematical explanations based on noticing equivalent ratios.

Emphasise that when all of the rectangles are aligned with the lower left angle matching, by increasing size:

- The diagonals of the rectangles fall into two sets: those that are scaled copies of the full sheet of paper and those that are scaled copies of the half sheet of paper.
- The diagonals of the rectangles that are scaled copies of one another match up.

Tell students that they are going to study a new kind of transformation (to be added to the list from previous work: translations, rotations, and reflections). This new kind of transformation makes scaled copies and is called an enlargement. An enlargement has a centre of enlargement(the common vertex for the rectangles in each stack) and a scale factor (4, for example, from rectangle E to rectangle A). Different choices of scale factor give scaled copies of different sizes: for example, rectangle C uses a scale factor of 2, applied to rectangle E.

1.3 Scaled Rectangles

Optional: 10 minutes

This activity continues to examine scaled copies of a rectangle via dividing a rectangle into smaller rectangles. In this activity, the focus is more on the scale factor and the language of scaled copies, emphasising the link with work students did earlier. Unlike in the previous task, there are no given dimensions for any of the rectangles. Students need to find the scale factor using their understanding of the meaning of scale factor and the fact that the rectangles are divided evenly.

Monitor how students reason about the scale factor. They could use the diagram to see how many times as long and wide one rectangle is compared to another. They could also use what they know about how the area of rectangles changes in scaled copies. Select students who use these approaches to share during the discussion.

Instructional Routines

- Co-Craft Questions

Launch

Tell students that they are going to examine a different set of rectangles and determine scale factors for pairs which are scaled copies of one another. Briefly, ask students to interpret what is meant by "evenly divided." (Rectangle R is cut exactly in half vertically and horizontally, and also one of its quadrants is cut exactly in thirds vertically and horizontally.) Students may want to use a ruler to validate their understanding of what "evenly divided" means.

Give students 5 minutes quiet work time followed by a whole-class discussion.

Engagement: Develop Effort and Persistence. Encourage and support opportunities for peer interactions. Prior to the whole-class discussion, invite students to share their work with a partner. Display sentence frames to support student conversation such as: "First, I _____ because ....", "I noticed _____ so I ....", "Why did you ...?", "I agree/disagree because ...."
Supports accessibility for: Language; Social-emotional skills Conversing, Writing: Co-Craft Questions. Before presenting the questions in this activity, display the diagram of the divided rectangles and ask students to write possible mathematical questions about the diagram. Invite students to compare the questions they generated with a partner before sharing the questions with the whole class. Listen for and amplify questions about whether two or more rectangles are scaled copies of one another. If no student asks whether two rectangles are scaled copies of one another, ask students to adapt a question to align with the learning goals of this lesson. Then reveal and ask students to work on the actual questions of the task. This routine will help develop students' meta-awareness of language as they generate questions about scaled rectangles.
Design Principle(s): Maximise meta-awareness

Student Task Statement

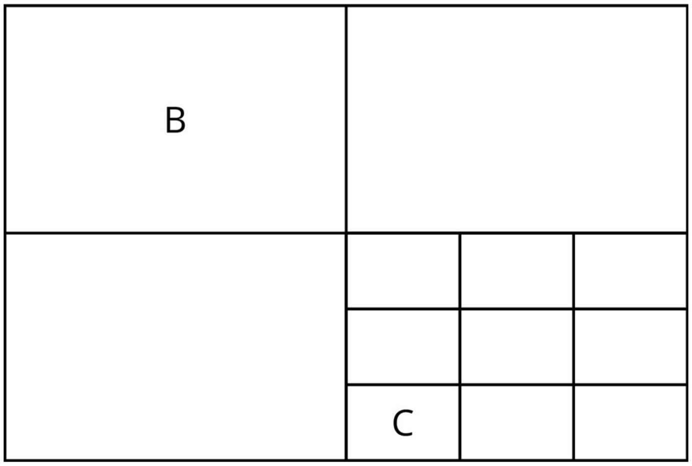
Here is a picture of rectangle R, which has been evenly divided into smaller rectangles. Two of the smaller rectangles are labelled B and C.

1. Is B a scaled copy of R? If so, what is the **scale factor**?

2. Is C a scaled copy of B? If so, what is the scale factor?

3. Is C a scaled copy of R? If so, what is the scale factor?

Rectangle R



Student Response

1. Yes, the length and width of rectangle B are each $\frac{1}{2}$ the length and width of rectangle R. The scale factor is $\frac{1}{2}$.

2. Yes, the length and width of rectangle C are each $\frac{1}{3}$ the length and width of rectangle B. The scale factor is $\frac{1}{3}$.

3. Yes, the length and width of rectangle C are each $\frac{1}{6}$ the length and width of rectangle R. The scale factor is $\frac{1}{6}$

Activity Synthesis

Ask selected students to share their solutions. Then ask these questions:

- "Why is rectangle B a scaled copy of rectangle R?" (The length and width in both cases have been multiplied by the same number because the rectangles are divided evenly.)
- "How are the scale factors from R to B and B to C related to the scale factor from R to C?" (The latter is the product of the former.)
- "Does the diagonal from top left to lower right of rectangle R go through opposite vertices of one rectangle of each size?" (Yes.)

Lesson Synthesis

In previous lessons, we have studied translations, rotations, and reflections. When we apply a sequence of these transformations to a shape, we change the shape's location and orientation in the plane but not its size. In this lesson, we began to study a new "move," which makes scaled copies of shapes (and hence can change their size!) This new move is called an *enlargement*. We will introduce a formal definition of enlargement in the next lesson and then we will investigate how shapes change when enlargements are allowed in addition to the previous transformations.

1.4 What is an Enlargement?

Cool Down: 5 minutes

Students will learn the formal definition of an enlargement in the next lesson. For now, they should just describe their current understanding of an enlargement and how it works. Students may think that enlargements only expand things because of its everyday meaning. In the next lesson, they will learn this is not always true.

Launch

Ask students to think about the rectangles they worked with to explain their understanding of enlargements and how they work.

Student Task Statement

In your own words, explain what an enlargement is.

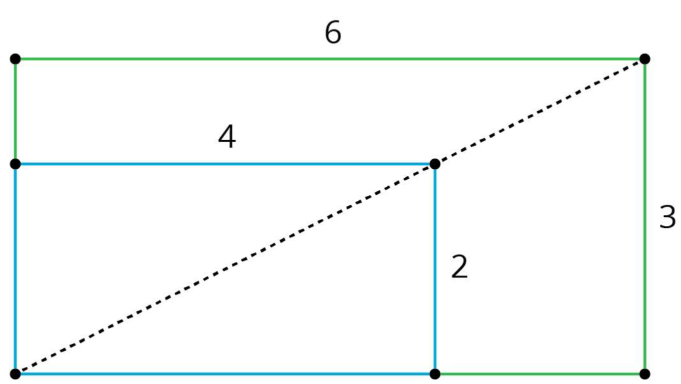
Student Response

Answers vary. Sample responses: An enlargement expands things. It makes scaled copies. It blows things up. It works by pushing things outward from a centre. An enlargement multiplies distances from the centre to ***change the size*** of the shape.

Student Lesson Summary

Scaled copies of rectangles have an interesting property. Can you see what it is?

Here, the larger rectangle is a scaled copy of the smaller one (with a scale factor of $\frac{3}{2}$). Notice how the diagonal of the large rectangle contains the diagonal of the smaller rectangle. This is the case for any two scaled copies of a rectangle if we line them up as shown. If two rectangles are *not* scaled copies of one another, then the diagonals do not match up. In this unit, we will investigate how to make scaled copies of a shape.



Glossary

- scale factor

Lesson 1 Practice Problems

1. **Problem 1 Statement**

   Rectangle A measures 12 cm by 3 cm. Rectangle B is a scaled copy of rectangle A. Select **all** of the measurement pairs that could be the dimensions of rectangle B.

   a. 6 cm by 1.5 cm

   b. 10 cm by 2 cm

   c. 13 cm by 4 cm

   d. 18 cm by 4.5 cm

   e. 80 cm by 20 cm

   Solution ["A", "D", "E"]

2. **Problem 2 Statement**

   Rectangle A has length 12 and width 8. Rectangle B has length 15 and width 10. Rectangle C has length 30 and width 15.

   a. Is rectangle A a scaled copy of rectangle B? If so, what is the scale factor?

   b. Is rectangle B a scaled copy of rectangle A? If so, what is the scale factor?

   c. Explain how you know that rectangle C is *not* a scaled copy of rectangle B.

   d. Is rectangle A a scaled copy of rectangle C? If so, what is the scale factor?

   Solution

   a. Yes, the scale factor is $\frac{4}{5}$.

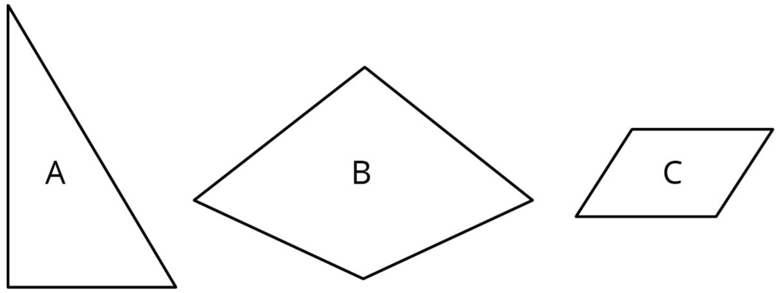
   b. Yes, the scale factor is $\frac{5}{4}$.

   c. Rectangle C's length is double that of rectangle B, but its width is not double.

   d. No.

3. **Problem 3 Statement**

   Here are three polygons.



a. Draw a scaled copy of polygon A with scale factor $\frac{1}{2}$.

b. Draw a scaled copy of polygon B with scale factor 2.

c. Draw a scaled copy of polygon C with scale factor $\frac{1}{4}$.

Solution

The scaled copy of polygon A should be a right triangle with each side half as long as the original.

The scaled copy of polygon B should be a quadrilateral with each side twice as long as the original.

The scaled copy of polygon C should be a parallelogram with each side one quarter the length of the original.

4. **Problem 4 Statement**

Which of these sets of angles could be the three angles in a triangle?

a. 40°, 50°, 60°

b. 50°, 60°, 70°

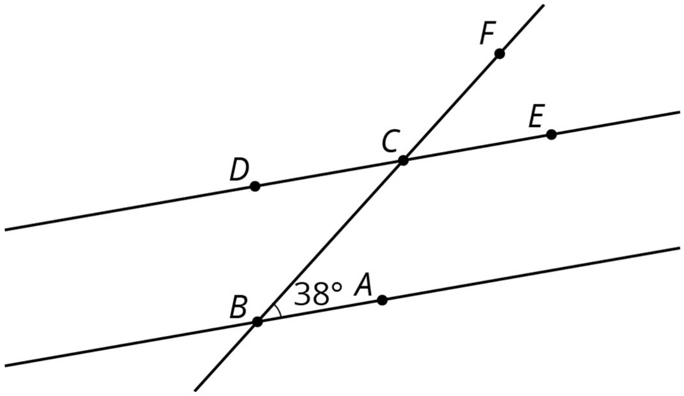
c. 60°, 70°, 80°

d. 70°, 80°, 90°

Solution B

5. **Problem 5 Statement**

In the picture lines AB and CD are parallel. Find the following angles. Explain your reasoning.



a. $\angle BCD$

b. $\angle ECF$

c. $\angle DCF$

Solution

a. 38 degrees. $\angle BCD$ and $\angle ABC$ are alternate angles for the parallel lines AB and CD cut by the transversal BC.

b. 38 degrees. $\angle ECF$ and $\angle BCD$ are a pair of vertically opposite angles.

c. 142 degrees. $\angle DCF$ and $\angle ECF$ are supplementary angles.