Lesson 9: Mean

Goals

- Comprehend the words "mean" and "average" as a measure of centre that summarises the data using a single number.
- Explain (using words and other representations) how to calculate the mean for a numerical/quantitative data set.
- Interpret diagrams that represent finding the mean as a process of levelling out the data to find a "fair share."

Learning Targets

- I can explain how the mean for a data set represents a "fair share."
- I can find the mean for a numerical/quantitative data set.

Lesson Narrative

In this lesson, students find and interpret the **mean** of a distribution as the amount each member of the group would get if everything is distributed equally. This is sometimes called the "levelling out" or the "fair share" interpretation of the mean. For a quantity that cannot actually be redistributed, like the weights of the dogs in a group, this interpretation translates into a thought experiment.

Suppose all of the dogs in a group had different weights and their combined weight was 200 pounds. The mean would be the weight of the dogs if all the dogs were replaced with the same number of identical dogs and the total weight was still 200 pounds.

Here students do not yet make an explicit connection between the mean and the idea of "typical," or between the mean and the centre of a distribution. These connections will be made in upcoming lessons.

Building On

- Use the four operations with whole numbers to solve problems.

Addressing

- Recognise that a measure of centre for a quantitative data set summarises all of its values with a single number, while a measure of variation describes how its values vary with a single number.
- Summarise and describe distributions.
- Giving quantitative measures of centre (median and/or mean) and variability (interquartile range), as well as describing any overall pattern and any striking deviations from the overall pattern with reference to the context in which the data were gathered.

Building Towards

- Recognise that a measure of centre for a quantitative data set summarises all of its values with a single number, while a measure of variation describes how its values vary with a single number.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- Compare and Connect
- Discussion Supports
- Think Pair Share

Required Materials

Multi-link cubes
Straightedges
A rigid edge that can be used for drawing line segments. Sometimes a ruler is okay to use as a straightedge, but sometimes it is preferable to use an unruled straightedge, like a blank index card.

Student Learning Goals

Let's explore the mean of a data set and what it tells us.

9.1 Close to Four

Warm Up: 5 minutes

The purpose of this warm-up is to prepare students to find the mean of a data set. While the goal of the activity is for students to create an expression with a value close to 4, the discussions should focus on the reasoning and strategies students used in creating their expression. Students should notice that they would get the same result if they divided the value of the entire expression in the numerator by 4 as they would if they divided each number in the numerator by 4 because there are 4 numbers in the numerator.

During the partner discussions, identify students with different strategies for creating an expression with a value of 4. Ask them to share during the whole-class discussion.

Instructional Routines

- Think Pair Share

Launch

Arrange students in groups of 2. Give students 1 minute of quiet work time, and then 2 minutes to share their response with a partner. Follow with a whole-class discussion. If students do not interpret the directions of "close to 4" to mean that their expression can have an exact value of 4, tell them it can be exactly 4.

Anticipated Misconceptions

Some students may think they need to use all of the digits from 0 to 9. Tell them that only 4 digits need to be used, although they are welcome to try finding a good solution using 2 digit numbers if they want.

Student Task Statement

Use the digits 0–9 to write an expression with a value as close as possible to 4. Each digit can be used only one time in the expression.

$(\square + \square + \square + \square) \div 4$

Student Response

Answers vary. Sample response: $(\boxed{2} + \boxed{8} + \boxed{5} + \boxed{1}) \div 4$. Since the expression should be close to 4, the numerator of the fraction should be close to 16.

Activity Synthesis

Poll the class on whether the value of their expression is exactly 4 or is close to 4. Ask selected students to share their strategy for creating an expression with a value of 4. Record and display their responses for all to see.

As students share their reasoning, consider asking some of the following questions:

- "How did you decide on the value of the numerator?"
- "How did the denominator affect your strategy?"
- "How might your strategy change if the denominator was a different number, say, 6 or 10?"
- "How might your strategy change if the numerator had more numbers or fewer numbers?"

9.2 Spread Out and Share

15 minutes (there is a digital version of this activity)

This activity introduces students to the concept of **mean** or **average** in terms of equal distribution or fair share. The two contexts chosen are simple and accessible, and include both discrete and continuous values. Diagrams are used to help students visualise the distribution of values into equal amounts.

The first set of problems (about cats in crates) can be made even more concrete by providing students with blocks or multi-link cubes that they can physically distribute into piles or containers. Students using the digital activities will engage with an applet that allows students to sort cats. For the second set of problems (about hours of work), students are prompted to draw two representations of the number of hours of work before and after they are redistributed, creating a visual representation of fair shares or quantities being levelled out.

As students work, identify those with very different ways of arranging cats into crates to obtain a mean of 6 cats. Also look for students who determine the redistributed work hours differently. For example, some students may do so by moving the number of hours bit by bit, from a server with the most hours to the one with the fewest hours, and continue to adjust until all servers have the same number. Others may add all the hours and divide the sum by the number of servers.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- Compare and Connect
- Think Pair Share

Launch

Arrange students in groups of 2. Provide access to straightedges. Also consider providing multi-link cubes for students who might want to use them to physically show redistribution of data values. If using the digital lesson, students will have access to an applet that will allow them to sort cats, multi-link cubes may not be necessary but can be provided.

Give students 3–4 minutes of quiet work time to complete the first set of questions and 1–2 minutes to share their responses with a partner. Since there are many possible correct responses to the question about the crates in a second room, consider asking students to convince their partner that the distribution that they came up with indeed has an average of 3 kittens per crate. Then, give students 4–5 min to work on the second set of questions together.

Representation: Develop Language and Symbols. Ensure access to virtual or concrete manipulatives to connect symbols to concrete objects or values. Provide students with blocks or multi-link cubes that they can physically distribute into piles or containers. *Supports accessibility for: Conceptual processing*

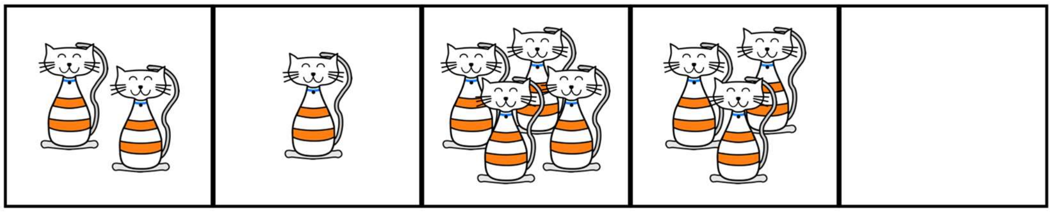
Anticipated Misconceptions

In the first room, to get each crate to have the same number of cats, some students might add new cats, not realising that to “distribute equally” means to rearrange and reallocate existing quantities, rather than adding new quantities. Clarify the meaning of the phrase for these students.

Some students may not recognise that the hours for the servers could be divided so as to not be whole numbers. For example, some may try to give 4 servers 6 hours and 1 server has 7 hours. In this case, the time spent working is still not really divided equally, so ask the student to think of dividing the hours among the servers more evenly if possible.

Student Task Statement

1. The kittens in a room at an animal shelter are placed in 5 crates.



   a. The manager of the shelter wants the kittens distributed equally among the crates. How might that be done? How many kittens will end up in each crate?

   b. The number of kittens in each crate after they are equally distributed is called the **mean** number of kittens per crate, or the **average** number of kittens per crate. Explain how the expression $10 \div 5$ is related to the mean.

   c. Another room in the shelter has 6 crates. No two crates has the same number of kittens, and there is a mean of 3 kittens per crate. Draw or describe at least two different arrangements of kittens that match this description.

2. Five servers were scheduled to work the number of hours shown. They decided to share the workload, so each one would work equal hours.

   server A: 3

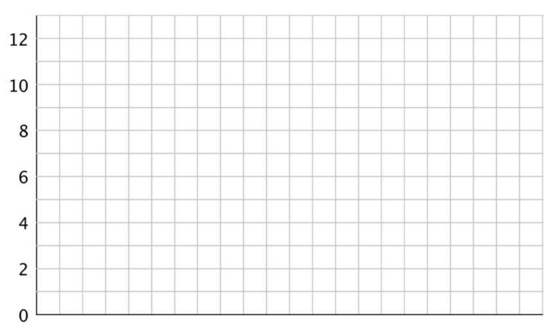
   server B: 6

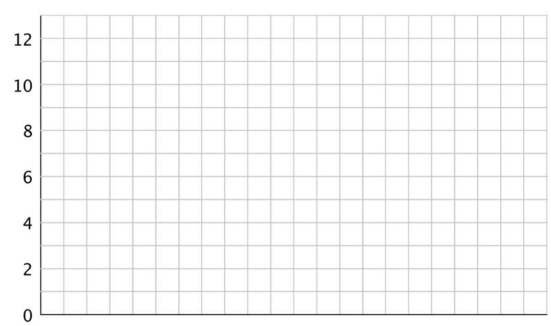
   server C: 11

   server D: 7

   server E: 4

   a. On the grid on the left, draw 5 bars whose heights represent the hours worked by servers A, B, C, D, and E.





b. Think about how you would rearrange the hours so that each server gets a fair share. Then, on the grid on the right, draw a new graph to represent the rearranged hours. Be prepared to explain your reasoning.

c. Based on your second drawing, what is the average or mean number of hours that the servers will work?

d. Explain why we can also find the mean by finding the value of the expression $31 \div 5$.

e. Which server will see the biggest change to work hours? Which server will see the least change?

Student Response

1.

a. Answers vary. Sample response: Add up the numbers of kittens (a total of 10) and divide that number by 5, which results in 2 kittens per crate.

b. The expression is the total number of kittens divided by the number of crates, which is the number of kittens in each crate after they are evenly distributed.

c. Answers vary. Sample response: The kittens could be distributed among the 6 crates in the following order: 2, 5, 1, 3, 0, 7, and 6, 2, 4, 5, 0, 1. Since there are 6 crates and an average of 3 kittens in each crates, there are a total of 18 kittens. Any distribution that has a total of 18 kittens will have a mean of 3 kittens in each crate.

2.

a.

b. The mean number of hours is 6.2 hours.

c. The expression shows all the hours being added and divided by 5, which gives us the fair share for each server. $\frac{(3+6+11+7+\ \ )}{5} = 6.2$

d. Server C will see the biggest change; their work hours will drop by close to 5 hours. Server B will barely see a difference; their work hours will increase only by $\frac{1}{5}$ of an hour or 12 minutes.

Are You Ready for More?

Server F, working 7 hours, offers to join the group of five servers, sharing their workload. If server F joins, will the mean number of hours worked increase or decrease? Explain how you know.

Student Response

Increase. Since the mean was 6.2 hours and Server F has 7 hours, it will increase everyone else's hours to even things out again.

Activity Synthesis

Invite several students with different arrangements of cats in the second room with 6 crates to share their solutions and how they know the mean number of cats for their solutions is 3. Make sure everyone understands that their arrangement is correct as long as it had a total of 18 kittens and 6 crates and no two crates have the same number of cats. Show that the correct arrangements could redistribute the 18 cats such that there are 3 cats per crate.

Then, select previously identified students to share how they found the redistributed work hours if the workers were to spread the workload equally. Start with students who reallocated the hours incrementally (from one server to another server) until the hours level out, and then those who added the work hours and dividing the sum by 5.

Students should see that the mean can be interpreted as what each member of the group would get if everything is distributed equally, without changing the sum of values.

Representing: Compare and Connect. After the selected students share their solutions, invite students to discuss "What is the same and what is different?" about the approaches. Call students' attention to the connections between approaches by asking "How is the approach 'divide the sum of work hours by 5' represented visually in the new bar graph?" This will help students make sense of and use different representations of the mean of a set of data. *Design Principle(s): Maximise meta-awareness; Support sense-making*

9.3 Getting to School

15 minutes

In this activity, students calculate the mean of a data set and interpret it in the context of the given situation. The first data set students see here has a dozen values, discouraging students from redistributing the values incrementally and encouraging them to use a more efficient method. In the second question, students analyse the values in data sets and use the structure to decide whether or not it makes sense that a given mean would match the data set.

As students work and discuss, notice the reasons they give for why the data sets in the second question could or could not be Tyler's data set. Identify students who recognise that the mean of a data set cannot be expected to be higher or lower than most of the values of the data set, and that a fair-share value would have a value that is roughly in the middle of data values.

Instructional Routines

- Discussion Supports
- Think Pair Share

Launch

Keep students in groups of 2. Give them 6–7 minutes of quiet work time, and then time to discuss their responses with their partner.

Representation: Internalise Comprehension. Activate or supply background knowledge about determining means of data sets. Allow students to use calculators to ensure inclusive participation in the activity.
Supports accessibility for: Memory; Conceptual processing

Student Task Statement

For the past 12 school days, Mai has recorded how long her bus rides to school take in minutes. The times she recorded are shown below.

9 8 6 9 10 7 6 12 9 8 10 8

1. Find the mean for Mai's data. Show your reasoning.
2. In this situation, what does the mean tell us about Mai's trip to school?
3. For 5 days, Tyler has recorded how long his walks to school take in minutes. The mean for his data is 11 minutes. Without calculating, predict if each of the data sets shown could be Tyler's. Explain your reasoning.
   - data set A: 11, 8, 7, 9, 8
   - data set B: 12, 7, 13, 9, 14
   - data set C: 11, 20, 6, 9, 10
   - data set D: 8, 10, 9, 11, 11
4. Determine which data set is Tyler's. Explain how you know.

Student Response

1. The mean of the data is 8.5 minutes. The mean is found by summing the minutes of travel and dividing by the number of rides: $(9 + 8 + 6 + 9 + 10 + 7 + 6 + 12 + 9 + 8 + 10 + 8) \div 12 = 8.5$

2. Answers vary. Sample response: The mean tells us that if the minutes of travel were all levelled out across the 12 days, Mai's trip to school would take 8.5 minutes each day.

3. Answers vary. Sample response: Data set A and data set D could not be Tyler's data, since all of the numbers in each set are either at or below 11. Data set B and data set C could be Tyler's data, since the numbers in each set are distributed around 11.

4. Data set B is Tyler's, since the mean of the numbers is $(11 + 20 + 6 + 9 + 10) \div 5 = 55 \div 5 = 11$.

Activity Synthesis

Select a couple of students to share how they found the mean of Mai's travel times. Poll the class briefly to see if others in the class found the mean the same way.

Then, focus the discussion on the second task and on what values could be reasonably expected of a data set with a particular mean. Ask students how they decided to rule out or keep certain sets of data as potentially belonging to Tyler. If not mentioned by students, highlight that the mean of a data set would be a value in the middle of the range of numbers in order for it to be a fair-share value.

Point out that, unlike hours of work or cats in crates, the times of travel here cannot actually be redistributed. The interpretation of mean translates into a thought experiment:

The mean is the travel time each day if all the travel times in the set were the same such that the combined travel time for this data set and the original data set was the same.

Speaking, Listening: Discussion Supports. Use this routine to support students' explanations of which data set is Tyler's. Invite students to use the sentence frame "Data set ___ is Tyler's because . . . ." Invite listeners to use the sentence frame "I agree/disagree, because . . . ." This will help students explain their own reasoning and make the sense of the reasoning of others.
Design Principle(s): Support sense-making; Optimise output (for justification

Lesson Synthesis

In this lesson, we look at finding the **mean** or the **average** of a numerical data set.

- "Suppose that a data set contains the amounts of money in five piggy banks. What would the mean of this data set tell us?"

- "Why might it make sense to think of the mean as a 'fair share?'"

- "How do we find the mean of a data set?"

- “How can we interpret the mean of the heights of students in a class?”

9.4 Finding Means

Cool Down: 5 minutes

Student Task Statement

1. Last week, the daily low temperatures for a city, in degrees Celsius, were 5, 8, 6, 5, 10, 7, and 1. What was the average low temperature? Show your reasoning.

2. The mean of four numbers is 7. Three of the numbers are 5, 7, and 7. What is the fourth number? Explain your reasoning.

Student Response

1. The mean low temperature was 6 degrees Celsius. The sum of the temperatures divided by the total number of recorded temperatures is $(5 + 8 + 6 + 5 + 10 + 7 + 1) \div 7 = 6$.

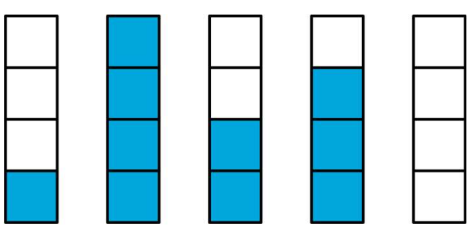
2. The fourth number is 9. The 4 numbers must be distributed evenly around 7. Since 2 of the numbers are 7, and the third number is two less than 7, the fourth number must be 2 more than 7.

Student Lesson Summary

Sometimes a general description of a distribution does not give enough information, and a more precise way to talk about centre or spread would be more useful. The **mean**, or **average**, is a number we can use to summarise a distribution.

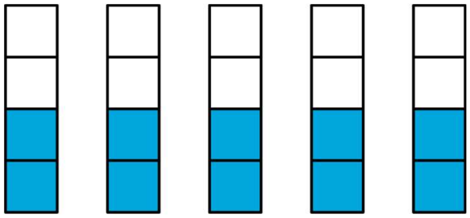
We can think about the mean in terms of “fair share” or “levelling out.” That is, a mean can be thought of as a number that each member of a group would have if all the data values were combined and distributed equally among the members.

For example, suppose there are 5 bottles which have the following amounts of water: 1 litre, 4 litres, 2 litres, 3 litres, and 0 litres.



To find the mean, first we add up all of the values. We can think of as putting all of the water together: $1 + 4 + 2 + 3 + 0 = 10$.

To find the "fair share," we divide the 10 litres equally into the 5 containers: $10 \div 5 = 2$.



Suppose the quiz scores of a student are 70, 90, 86, and 94. We can find the mean (or average) score by finding the sum of the scores ($70 + 90 + 86 + 94 = 340$) and dividing the sum by four ($340 \div 4 = 85$). We can then say that the student scored, on average, 85 points on the quizzes.

In general, to find the mean of a data set with n values, we add all of the values and divide the sum by n.

Glossary

- average
- mean

Lesson 9 Practice Problems

Problem 1 Statement

A preschool teacher is rearranging four boxes of playing blocks so that each box contains an equal number of blocks. Currently box 1 has 32 blocks, box 2 has 18, box 3 has 41, and box 4 has 9.

Select **all** the ways he could make each box have the same number of blocks.

a. Remove all the blocks and make four equal piles of 25, then put each pile in one of the boxes.

b. Remove 7 blocks from box 1 and place them in box 2.

c. Remove 21 blocks from box 3 and place them in box 4.

d. Remove 7 blocks from box 1 and place them in box 2, and remove 21 blocks from box 3 and place them in box 4.

e. Remove 7 blocks from box 1 and place them in box 2, and remove 16 blocks from box 3 and place them in box 4.

Solution ["A", "E"]

Problem 2 Statement

In a round of mini-golf, Clare records the number of strokes it takes to hit the ball into the hole of each green.

2 3 1 4 5 2 3 4 3

She said that, if she redistributed the strokes on different greens, she could tell that her mean number of strokes per hole is 3. Explain how Clare is correct.

Solution

Answers vary. Sample explanation: For both of the greens where she got 4 strokes, moving 1 stroke to the two greens where she got 2 strokes means that all 4 four of those greens now take 3 strokes. Likewise, moving 2 strokes from the green where it took her 5 strokes to the green where she got 1 stroke would also mean 3 strokes for each green.

Problem 3 Statement

Three year 7 classes raised £25.50, £49.75, and £37.25 for their classroom libraries. They agreed to share the money raised equally. What is each class's equal share? Explain or show your reasoning.

Solution

£37.50. Explanations vary. Sample explanation: The total raised is £112.50, and one-third of that is £37.50.

Problem 4 Statement

In her English class, Mai's teacher gives 4 quizzes each worth 5 points. After 3 quizzes, she has the scores 4, 3, and 4. What does she need to get on the last quiz to have a mean score of 4? Explain or show your reasoning.

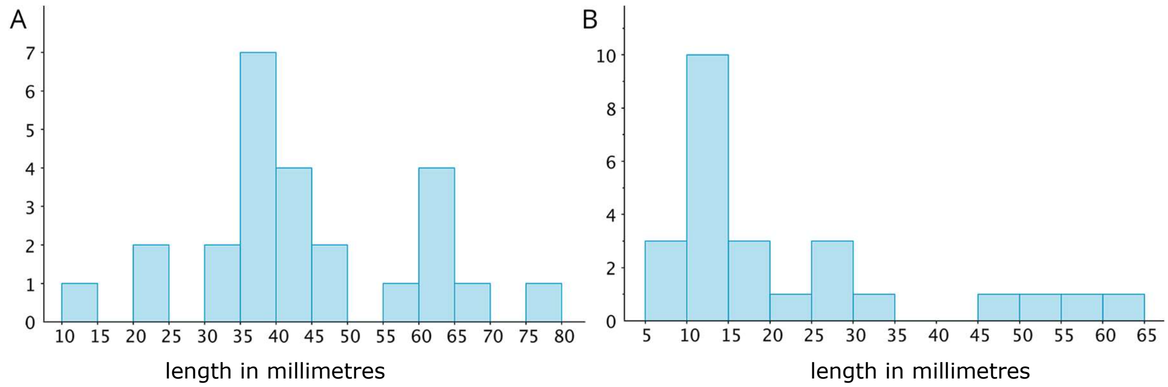
Solution

5. Explanations vary. Sample explanation: To get a mean of 4, one point needs to be redistributed to the score of 3, so the last quiz must be a 5 so that it can share one point and still be at the mean itself.

Problem 5 Statement

An earthworm farmer examined two containers of a certain species of earthworms so that he could learn about their lengths. He measured 25 earthworms in each container and recorded their lengths in millimetres.

Here are histograms of the lengths for each container.



a. Which container tends to have longer worms than the other container?

b. For which container would 15 millimetres be a reasonable description of a typical length of the worms in the container?

c. If length is related to age, which container had the most young worms?

Solution

a. Container A

b. Container B

c. Container B

Problem 6 Statement

Diego thinks that $x = 3$ is a solution to the equation $x^2 = 16$. Do you agree? Explain or show your reasoning.

Solution

No. Explanations vary. Sample explanation: I disagree with Diego. I tried using 3 for x in the equation, but $3^2 = 9$, not 16. Another sample explanation: I disagree with Diego. I know that $4^2 = 16$, so it cannot be true that $3^2 = 16$.