Lesson 6: Similarity

Goals

- Comprehend that the phrase "similar shapes" (in written and spoken language) means there is a sequence of translations, rotations, reflections, and enlargements that takes one shape to the other.
- Justify (orally) the similarity of two shapes using a sequence of transformations that takes one shape to the other.

Learning Targets

- I can apply a sequence of transformations to one shape to get a similar shape.
- I can use a sequence of transformations to explain why two shapes are similar.

Lesson Narrative

In the previous unit, students saw that two shapes are congruent when there is a sequence of translations, rotations, and reflections that takes one shape to another. Now enlargements, studied in previous lessons, are added to the possible set of "moves" taking one shape to another. Two shapes are **similar** if there is a sequence of translations, rotations, reflections, and enlargements that takes one shape to the other. When two shapes are similar, there are always many different sequences that show that they are similar. One method is to apply an enlargement to one shape so that the corresponding shapes are congruent. Then a sequence of translations, rotations and reflections will finish taking one shape to the other. Alternatively, we could translate one pair of corresponding vertices together, apply rotations and reflections to adjust the orientations, and then conclude with an enlargement so that they match.

In future lessons, students will learn shortcuts for some polygons (including all triangles), but in this lesson they focus on the definition of similarity in terms of transformations. They will see that two enlargements with the same scale factor but different centres differ by a translation. They will also study how transformations from polygon A to polygon B can be reversed to take polygon B to polygon A.

Building On

- Apply and extend previous understandings of operations with fractions to add, subtract, multiply, and divide rational numbers.
- Understand that a two-dimensional shape is congruent to another if the second can be obtained from the first by a sequence of rotations, reflections, and translations; given two congruent shapes, describe a sequence that exhibits the congruence between them.

Addressing

- Understand that a two-dimensional shape is congruent to another if the second can be obtained from the first by a sequence of rotations, reflections, and translations; given two congruent shapes, describe a sequence that exhibits the congruence between them.
- Understand that a two-dimensional shape is similar to another if the second can be obtained from the first by a sequence of rotations, reflections, translations, and enlargements; given two similar two-dimensional shapes, describe a sequence that exhibits the similarity between them.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- Stronger and Clearer Each Time
- Clarify, Critique, Correct
- Compare and Connect

Required Materials

Geometry toolkits

tracing paper, graph paper, coloured pencils, scissors, and an index card to use as a straightedge or to mark right angles, plus a ruler and protractor. Clear protractors with no holes and with radial lines printed on them are recommended.

Pre-printed slips, cut from copies of the blackline master

Methods for translations and enlargements Partner A Translate from A to D	Methods for translations and enlargements Partner B Translate from D to A
Methods for translations and enlargements Partner A Translate from P to D	Methods for translations and enlargements Partner B Translate from D to P
Methods for translations and enlargements Partner A Enlarge using centre A and scale factor 3	Methods for translations and enlargements Partner B Enlarge using centre A and scale factor $\frac{1}{3}$

Methods for translations and enlargements Partner A Enlarge using centre P and scale factor 3	Methods for translations and enlargements Partner B Enlarge using centre D and scale factor $\frac{1}{3}$
Methods for translations and enlargements Partner A Enlarge using centre D and scale factor 3	Methods for translations and enlargements Partner B Enlarge using centre P and scale factor $\frac{1}{3}$

Required Preparation

If you decide to do the optional "Methods for Translations and Enlargements" activity, print and cut out 1 set of cards for every 2 students.

Student Learning Goals

Let's explore similar shapes.

6.1 Equivalent Expressions

Warm Up: 5 minutes

This warm-up prompts students to use what they know about operations and properties of operations to create related expressions. While many warm-ups encourage students to work mentally and verbally, students will write their responses to this prompt. Since many different responses are possible, the task is accessible to all students and provides an opportunity to hear how each student reasons about the operations. Some different ideas that may emerge are:

- commutative property
- distributive property
- inverse operations
- adjusting factors (for example, doubling and halving)

Examples of each are given in the student response section. Students are not expected to use these terms, but highlight the terms if students do use them.

Launch

Arrange students in groups of 2. Tell students they are writing a list of several expressions equivalent to $10(2 + 3) - 8 \times 3$. Give students 2 minutes of quiet think time followed by 1 minute to discuss their responses with a partner.

Student Task Statement

Use what you know about operations and their properties to write three expressions equivalent to the expression shown.

$10(2 + 3) - 8 \times 3$

Student Response

Answers vary. Possible responses:

- commutative property: $10(3 + 2) - 8 \times 3$ or $-8 \times 3 + 10(2 + 3)$
- distributive property: $10 \times 2 + 10 \times 3 - 8 \times 3$
- inverse operations: $10(2 + 3) + -8 \times 3$
- associative property: $10(2 + 3) - 16 \times 1.5$

Activity Synthesis

Much of the discussion takes place between partners. Ask students to share any expressions that they aren't sure about, but try to resolve these and move on quickly.

6.2 Similarity Transformations (Part 1)

20 minutes (there is a digital version of this activity)

In this activity, students learn that two shapes are *similar* when there is a sequence of translations, reflections, rotations and enlargements that takes one shape to the other. Students practise discovering these sequences for two pairs of shapes.

When two shapes are similar but not congruent, the sequence of steps showing the similarity usually has a single enlargement and then the rest of the steps are translations, rotations and reflections. The enlargement can come at any time. It does not matter which shape you start with. An important thing for students to notice in this activity is that there is more than one sequence of transformations that show two shapes are similar. Monitor for students who insert an enlargement at different places in the sequence. Also monitor for how students find the scale factor for the hexagons.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- Stronger and Clearer Each Time

Launch

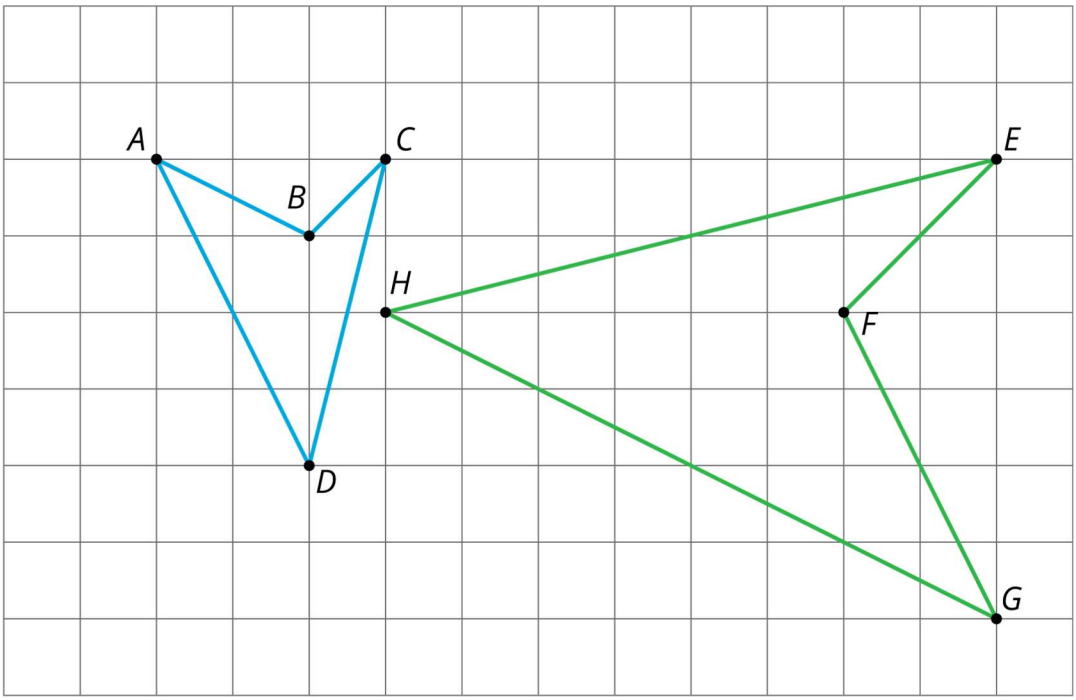
Before beginning this task, define what it means for two shapes to be *similar*.

Tell students: We have talked about how one shape can be a scaled copy of another. This relationship goes in both directions. For example, if triangle DEF is a scaled copy of triangle

ABC with scale factor of 2 then triangle ABC is a scaled copy of triangle DEF with scale factor $\frac{1}{2}$. We have learned that the transformation that creates scaled copies is called an enlargement.

We say that triangles ABC and DEF are **similar**. The previous unit explored how translations, rotations, and reflections define congruent shapes. The inclusion of enlargements can change the size of the shape as well as its location and orientation.

We will start our investigation of similar shapes by identifying sequences of translations, rotations, reflections, and enlargements that show two shapes are similar. Demonstrate using this example.



There are many methods to make this work. Explain at least two. First, identify the corresponding parts. Then come up with a plan to take one shape to the other. Ensure students understand, through demonstration with this example, that the work of showing two shapes are similar requires communicating the details of each transformation in the sequence with enough precision. Some sample methods:

1. Method 1 ($ABCD$ to $GFEH$: Enlarge, Translate, Rotate, Reflect)
    a. Enlarge using D as the centre with scale factor 2.
    b. Translate D to H
    c. Rotate using H as the centre clockwise by 90 degrees
    d. Reflect using the line that contains H and F.

2. Method 2 ($ABCD$ to $GFEH$: Reflect, Translate, Rotate, Enlarge)
    a. Reflect using the line that contains D and B.
    b. Translate D to H.
    c. Rotate using H as the centre clockwise by 90 degrees
    d. Enlarge using H as the centre with a scale factor of 2.
3. Method 3 ($ABCD$ to $GFEH$: Translate, Rotate, Reflect, Enlarge)
    a. Translate B to F.
    b. Rotate using F as the centre clockwise by 90 degrees.
    c. Reflect using the line that contains F and H.
    d. Enlarge using F as the centre with scale factor 2.

These arguments can also be applied to shapes that are not on a grid: the grid helps to identify directions and distances of translation, 90 degree angles of rotation, and horizontal and vertical lines of reflection.

If using the print version, provide access to geometry toolkits. If using the digital version, remind students of the meaning and functionality of each transformation tool, as necessary.

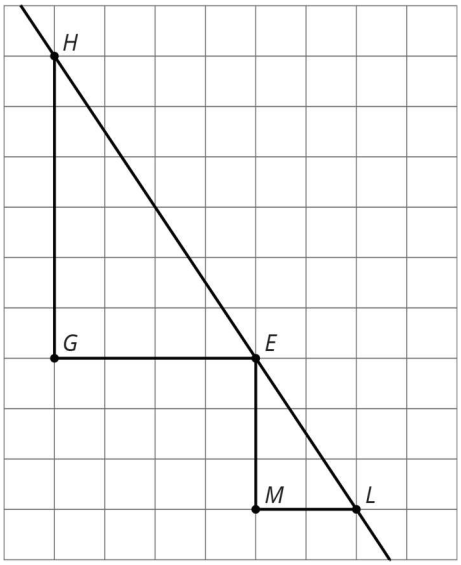
Representation: Develop Language and Symbols. Create a display of important terms and vocabulary. Include the following term and maintain the display for reference throughout the unit: similar. On this display, include the step-by-step instructions of at least 2 of the 3 given methods for creating similar polygons using transformations.
Supports accessibility for: Memory; Language

Anticipated Misconceptions

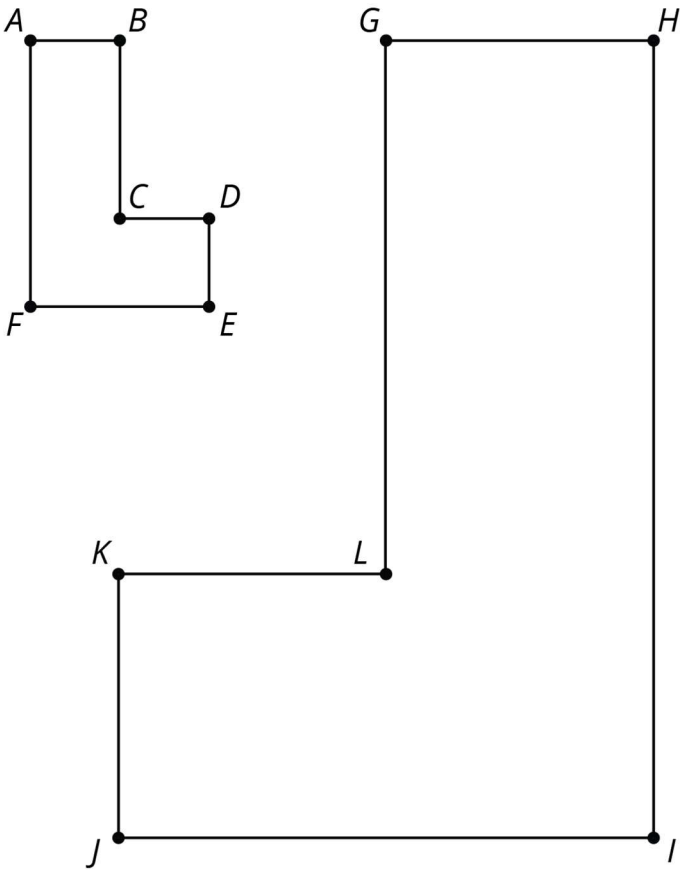
For the second problem, students may need encouragement to experiment moving the shapes (using tracing paper for example). If they get stuck finding the scale factor, tell them that they can approximate by measuring sides of the two shapes.

Student Task Statement

1. Triangle EGH and triangle LME are **similar**. Find a sequence of translations, rotations, reflections, and enlargements that shows this.



2. Hexagon $ABCDEF$ and hexagon $HGLKJI$ are similar. Find a sequence of translations, rotations, reflections, and enlargements that shows this.



Student Response

1. Answers vary. Sample response:

a. Begin with triangle LME

b. Translate L to E (2 units left, 3 units up)

c. Enlarge using E as the centre with scale factor 2

2. Answers vary. Sample response:

a. Begin with shape $ABCDEF$

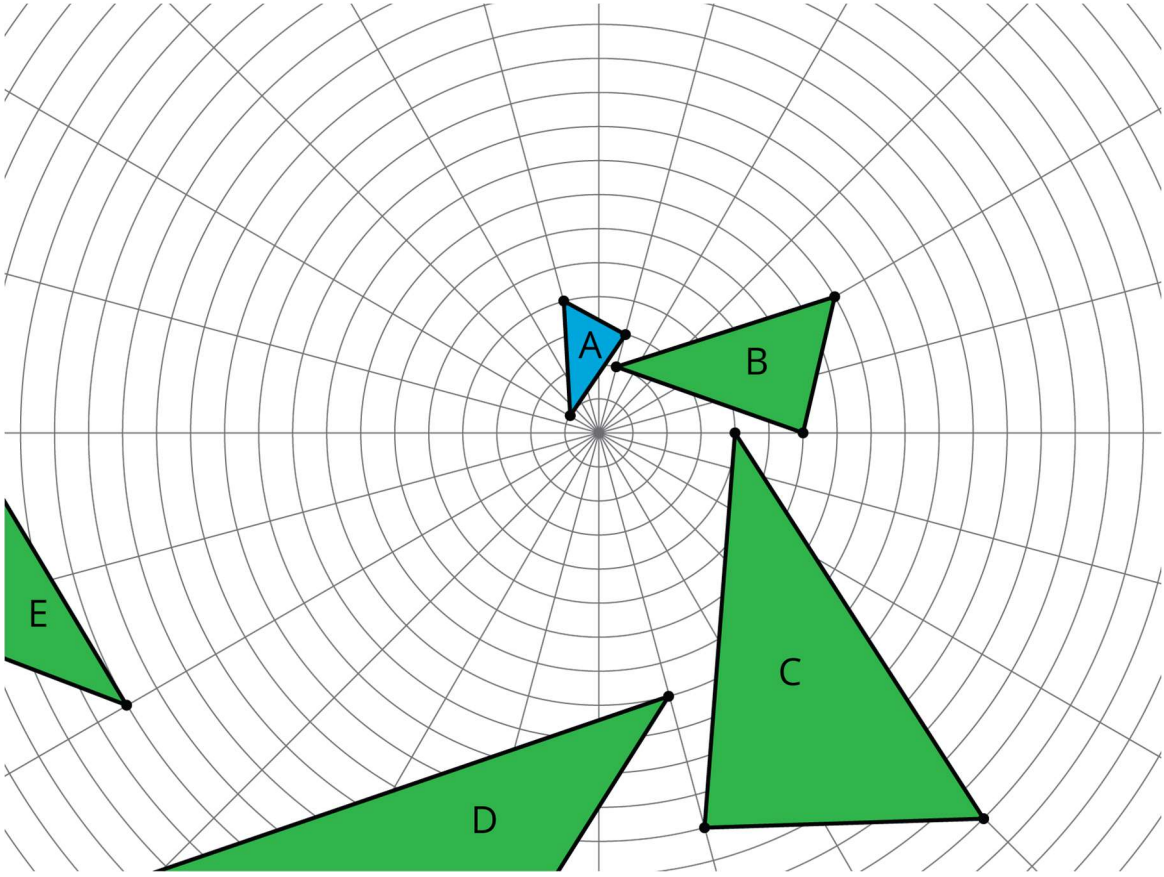
b. Reflect using the line that contains A and F

c. Translate F to I

d. Enlarge using I as the centre with scale factor 3

Are You Ready for More?

The same sequence of transformations takes triangle A to triangle B, takes triangle B to triangle C, and so on. Describe a sequence of transformations with this property.



Student Response

Answers vary. Sample response: Enlarge, from the centre of the circular grid, with scale factor 2, then rotate clockwise 75 degrees.

Activity Synthesis

Select students to give a variety of solutions for the different problems. Point out that there are multiple ways to do each pair and any valid sequence is allowed. Ensure that students communicate each transformation in the sequence in sufficient detail.

Writing, Speaking, Listening: Stronger and Clearer Each Time. After students have determined the sequence of transformations that shows the polygons are similar, ask students to write a detailed sequence of the transformations on their paper. Ask each student to meet with 2–3 other partners in a row for feedback. Provide students with prompts for feedback that will help them strengthen their ideas and clarify their language (e.g., "How did you know to translate point L to point E?", and "How did you know to enlarge the polygon by a scale factor of 3?", etc.). Students can borrow ideas and language from each partner to refine and clarify their original explanation. This will help students refine their own explanation and learn about other ways to show polygons are similar using a sequence of transformations.
Design Principles(s): Optimise output (for explanation); Maximise meta-awareness

6.3 Similarity Transformations (Part 2)

10 minutes

This activity helps students visualise what happens to shapes under different kinds of transformations. Students practise identifying which transformations might be used in the sequence of translations, rotations, reflections, and enlargements in order to show shapes are similar. By recognising patterns in the image results after using certain transformations, students may be able to apply this to finding transformations for other problems. Students should pay special attention to see the connection between the orientations of the original shape and the resulting images after transformation.

Encourage students to draw rough sketches though it is also ok to use tracing paper, for example, to execute the translations, rotations and reflections. For the enlargements, however, it could become time consuming to choose an explicit scale factor and measure carefully. Make sure, after students have worked on the first problem, to show some examples of sketches that are not exact but capture the main features of the shape.

Instructional Routines

- Compare and Connect

Launch

The shape in this task is intended to resemble a hand with all of the fingers together and the thumb sticking out. Encourage students to "sketch" the resulting images for this task

(or make tracing paper available, indicating that the images do not need to be exact). They do not need to make the side lengths, angles, etc. perfect, but it should be clear where the corresponding parts of the image are and whether it is larger or smaller than the original. Other activities in this unit will ask students to be precise in their use of transformations, but the goal of this activity is to get an idea of how the different transformations affect a shape's image.

Select a few good examples of student work (including some that have been sketched free hand) to share with the class after problem 1, in order to clearly communicate the expectation for level of precision.

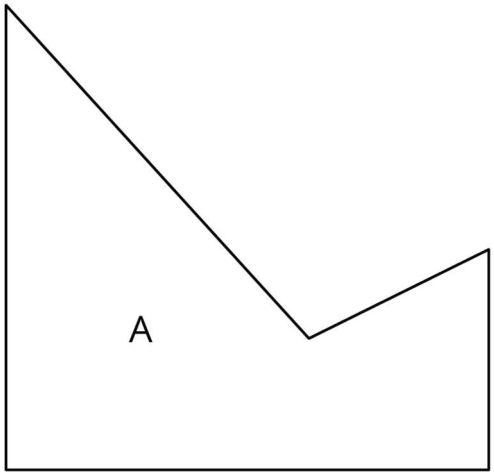
Action and Expression: Internalise Executive Functions. To support development of organisational skills, check in with students within the first 2–3 minutes of work time. Check to make sure students are not using exact or precise measurements, as this activity only requires a sketch. After the first problem, invite a few students to think aloud and share their sketches to guide the rest of the individual work time.
Supports accessibility for: Memory; Organisation Speaking, Listening: Compare and Connect. Ask students to prepare a visual display of their shapes that are similar to shape A. As students investigate each other's work, ask students to share what transformations are especially clear in the display of similar shapes. Listen for and amplify any comments about what might make the transformations clearer in the display. Then encourage students to make connections between the words "translation," "rotation," "reflection," and "enlargement" and how they affect the shape. Listen for and amplify language students use to describe what happens to shapes under different kinds of transformations. This will foster students' meta-awareness and support constructive conversations as they compare images of the same shape and make connections between transformations and their effects on shapes.
Design Principle(s): Cultivate conversation; Maximise meta-awareness

Anticipated Misconceptions

If students choose an exact scale factor, or measure the exact angle sizes, explain that precise measurements are not needed in this task. At this point, they are just sketching similar shapes.

Student Task Statement

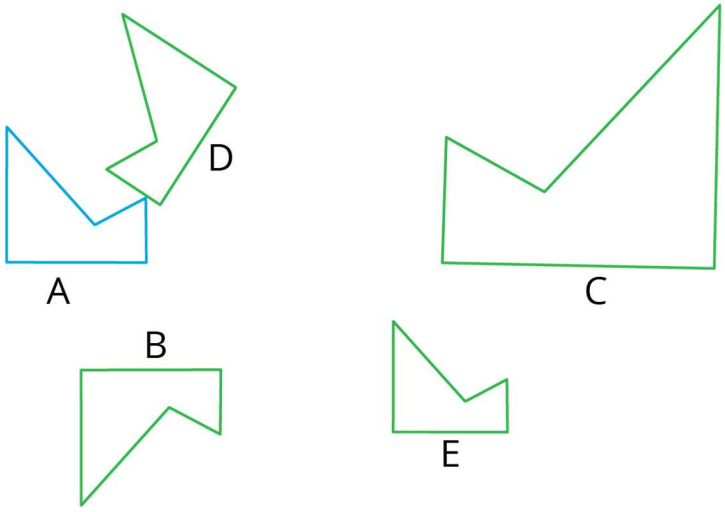
Sketch shapes similar to shape A that use only the transformations listed to show similarity.



1. A translation and a reflection. Label your sketch diagram B.
   Pause here so your teacher can review your work.
2. A reflection and an enlargement with scale factor greater than 1. Label your sketch diagram C.
3. A rotation and a reflection. Label your sketch diagram D.
4. An enlargement with scale factor less than 1 and a translation. Label your sketch diagram E.

Student Response

Answers vary. Sample responses:



Activity Synthesis

Select students to display their answers for each of the questions. Ask students to notice things the answers have in common so that they can make connections to the types of transformations that might be useful in showing that two shapes are similar.

Assuming that the rotations are not through an angle that is a multiple of 360° and that the translations have a non-zero horizontal or vertical part, point out that:

- Enlargements will create larger or smaller copies depending on the scale factor as seen in previous lessons.
- Translations will slide the shape in some direction.
- Rotations will "tilt" or "turn" the shape.
- Reflections will change the handedness so that the resulting image will look like the back of a right hand instead of the back of a left hand as in the original image.

6.4 Methods for Translations and Enlargements

Optional: 10 minutes

The purpose of this task is for students to practise showing that two shapes are similar using only a few pre-determined translations, rotations and reflections and enlargements. Some students will start with triangle ABC and take this to triangle DEF while other start with DEF and take this to ABC. While there is flexibility in either direction, one way of getting from DEF to ABC is to "undo" the moves that take ABC to DEF.

Monitor for students who use different centres of enlargement in their sequence, particularly as the first step in the sequence. Invite these students to share, highlighting the fact that two enlargements with the same scale factor but different centres differ by a translation. Also monitor for students whose sequences of translations, rotations and reflections and enlargements are the same but in the opposite order, one set taking ABC to DEF and the opposite taking DEF back to ABC. Select these students to share this important observation during the discussion.

Instructional Routines

- Clarify, Critique, Correct

Launch

Again, remind students that two shapes are similar if there is a sequence of translations, rotations, reflections, and enlargements that takes one shape to another. Tell them that they need to find at least one way to show that triangle ABC and triangle DEF are similar using only the transformations they are given on their cards.

Arrange students in groups of 2. Give each group one complete set of cards.

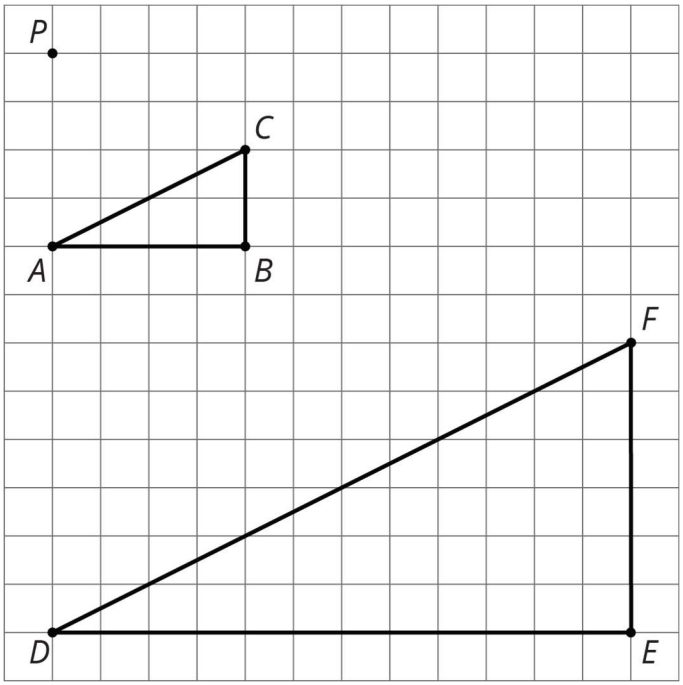
Engagement: Develop Effort and Persistence. Encourage and support opportunities for peer interactions. Invite students to talk about their ideas with a partner before writing them down. Display sentence frames to support students when they explain their strategy. For example, "One thing that is the same is…."; "One thing that is different is…."; and "Another strategy to get the same result is…."
Supports accessibility for: Language; Social-emotional skills

Anticipated Misconceptions

Students might think that is it necessary to perform transformations in the same order or that one particular point needs to be the centre. If partners choose the same methods, prompt them to try it another way that will have the same end result.

Student Task Statement

Your teacher will give you a set of five cards and your partner a different set of five cards. Using only the cards you were given, find at least one way to show that triangle ABC and triangle DEF are similar. Compare your method with your partner's method. What is the same about your methods? What is different?



Student Response

Partner A can:

1. Enlarge triangle ABC using centre point P and scale factor 3.
2. Enlarge using centre point A and scale factor 3 followed by translating from A to D.

3. Translate from A to D followed by enlarging using centre D and scale factor 3.

Partner B can:

1. Enlarge triangle DEF using centre point P and scale factor $\frac{1}{3}$.

2. Enlarge using centre point D and scale factor $\frac{1}{3}$ followed by translating from D to A.

3. Translate from D to A followed by enlarging using centre A and scale factor $\frac{1}{3}$.

Activity Synthesis

Invite selected students to share, highlighting methods of moving ABC to DEF and DEF to ABC which are "opposite" of one another, for example enlargements with centre P and reciprocal scale factors.

As students share their responses, highlight these points:

- The scale factors for the enlargements are reciprocals regardless of when the enlargements are done in the sequence.
- If used, the translations are inverses of each other (eg "Translate A to D" instead of "Translate D to A").
- Enlargements with different centres but the same scale factor produce congruent shapes that differ by a translation.
- The order in which transformations are applied can influence the result.

One important conclusion (the third bullet point) is that when you are showing that two shapes are similar, you can pick any point as the centre of enlargement if you know the scale factor, because you can always adjust the position using a translation.

Reading, Writing, Speaking: Clarify, Critique, Correct. Before students share their sequence of transformations to show that triangle ABC and triangle DEF are similar, present an incorrect sequence of transformations. For example, "Translate from A to D. Then enlarge using centre P and scale factor 3." Ask students to identify the error, critique the reasoning, and write a correct explanation. As students discuss in partners, listen for students who clarify the meaning of the centre of enlargement. Prompt students to share their critiques and corrected explanations with the class. Listen for and amplify the language students use to describe what happens when P is the centre of enlargement and explain why D should be the centre of enlargement. This routine will engage students in meta-awareness as they clarify how the centre of enlargement affects the enlarged shape.
Design Principle(s): Optimise output (for explanation); Maximise meta-awareness

Lesson Synthesis

Review the definition of similar shapes and any important insights that arose during the lesson. Insights to highlight:

Two shapes are similar if there is a sequence of translations, rotations, reflections, and enlargements that maps one to the other. Scaled copies of shapes are all examples of similar shapes. In this lesson, we found transformations that showed that two shapes were similar.

We saw that there is more than one sequence of transformations that shows two shapes are similar. One way to think about it is that you need to use an enlargement to make corresponding side lengths the same size. The shapes will be congruent after this step. Once you do that, you just need to find a sequence of translations, rotations and reflections that align the congruent shapes. You can also do it the other way, by bringing the shapes into alignment and then enlarging one to match up with the other.

Add the term *similar* along with a definition and example to your classroom display such as a word wall or anchor chart.

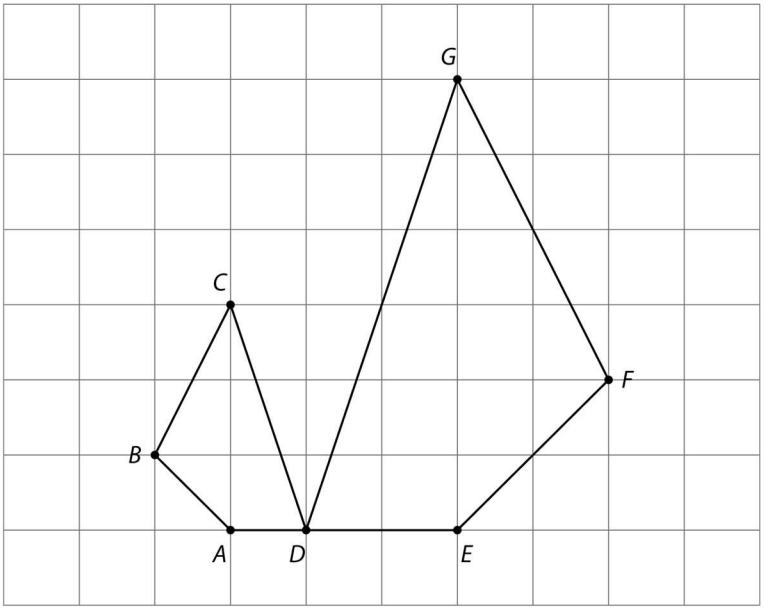
6.5 Showing Similarity

Cool Down: 5 minutes

Students analyse and correct a proposed sequence of translations, rotations and reflections and enlargements to show that two shapes are similar.

Student Task Statement

Elena gives the following sequence of transformations to show that the two shapes are similar by transforming $ABCD$ into $EFGD$.



1. Enlarge using centre D and scale factor 2.

2. Reflect using the line AE.

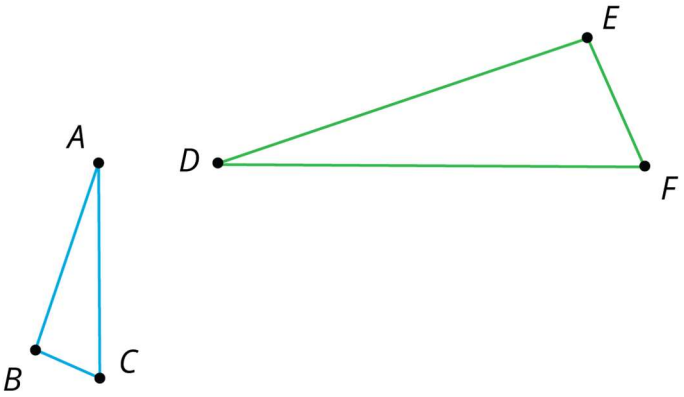
Is Elena's method correct? If not, explain how you could fix it.

Student Response

The shapes are similar, but the transformations do not take $ABCD$ to $EFGD$. After enlarging $ABCD$ using D as the centre with a scale factor of 2, Elena can reflect in the vertical line through D rather than the horizontal line.

Student Lesson Summary

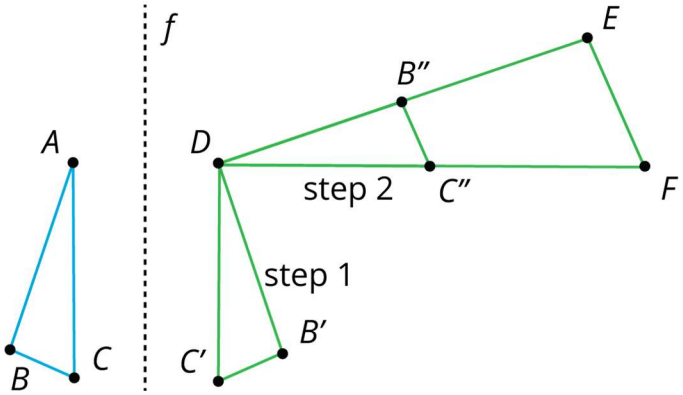
Let's show that triangle ABC is similar to triangle DEF:



Two shapes are **similar** if one shape can be transformed into the other by a sequence of translations, rotations, reflections, and enlargements. There are many correct sequences of transformations, but we only need to describe one to show that two shapes are similar.

One way to get from ABC to DEF follows these steps:

- step 1: reflect across line f
- step 2: rotate 90° anti-clockwise around D
- step 3: enlarge with centre D and scale factor 2



Another way would be to enlarge triangle ABC by a scale factor of 2 with centre of enlargement A, then translate A to D, then reflect in a vertical line through D, and finally rotate it so it matches up with triangle DEF. What steps would you choose to show the two triangles are similar?

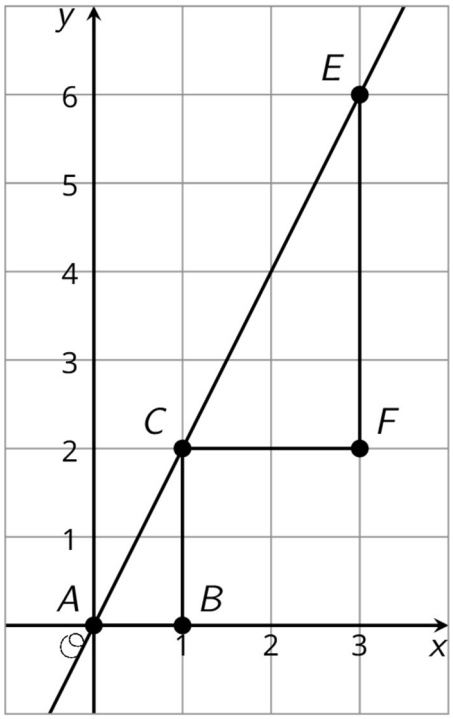
Glossary

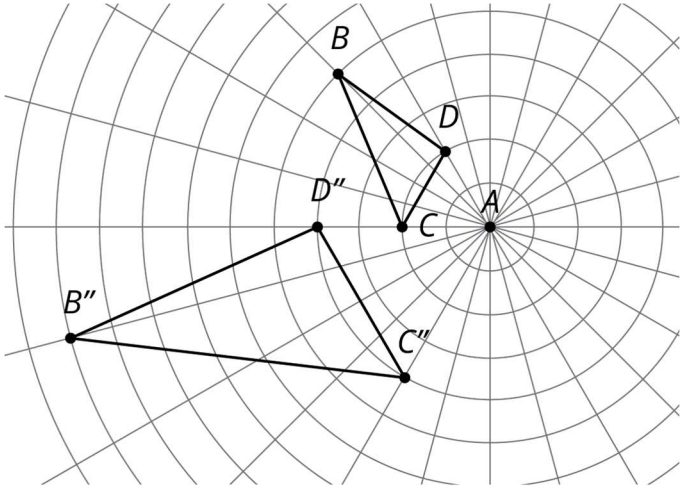
- similar

Lesson 6 Practice Problems

1. **Problem 1 Statement**

   Each diagram has a pair of shapes, one larger than the other. For each pair, show that the two shapes are similar by identifying a sequence of translations, rotations, reflections, and enlargements that takes the smaller shape to the larger one.





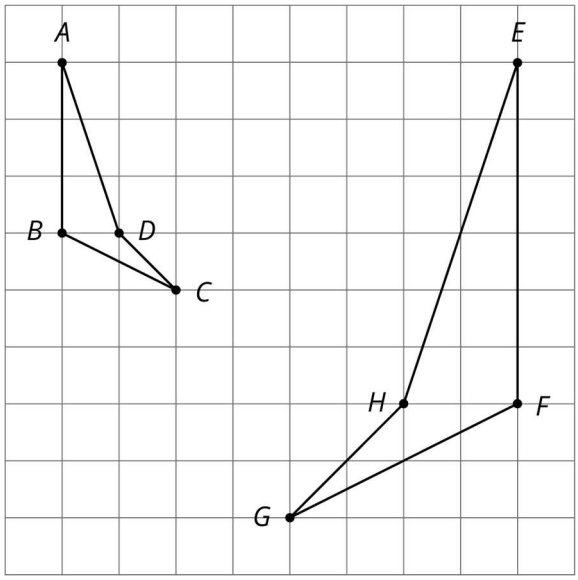
Solution

- Translate A to C, and then enlarge with centre A by a factor of 2.
- Rotate 60° anti-clockwise with centre A, and then enlarge using a scale factor of 2 centred at A.

2. **Problem 2 Statement**

Here are two similar polygons.

Measure the side lengths and angles of each polygon. What do you notice?

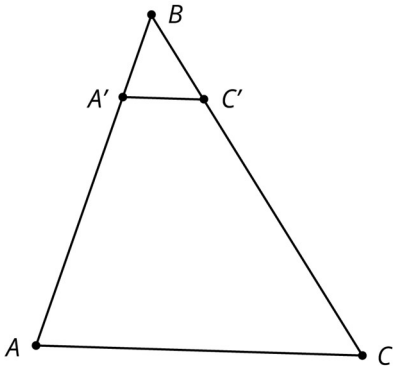


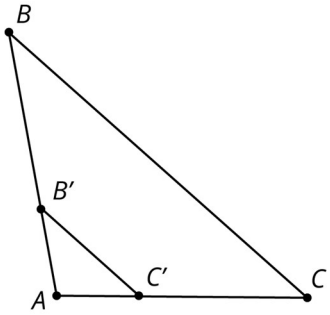
Solution

Answers vary. Sample response: Corresponding side lengths in the larger polygon are double the side lengths of the smaller polygon, while corresponding angles all have the same size.

3. **Problem 3 Statement**

Each shape shows a pair of similar triangles, one contained in the other. For each pair, describe a point and a scale factor to use for an enlargement moving the larger triangle to the smaller one. Use a measurement tool to find the scale factor.





Solution

Centre of enlargement: B, scale factor: $\frac{1}{4}$; centre of enlargement: A, scale factor: $\frac{1}{3}$