Lesson 12: Negative rates

Goals

- Apply operations with directed numbers to solve problems involving constant rates, and explain (orally) the solution method.
- Explain (orally and in writing) how directed numbers can be used to represent situations involving constant rates.
- Write an equation of the form $y = -kx$ to represent a situation that involves descending at a constant rate.

Learning Targets

- I can solve problems that involve multiplying and dividing rational numbers.
- I can solve problems that involve negative rates.

Lesson Narrative

The purpose of this lesson is to introduce students to negative rates of change, which will become important when they start learning about linear functions in later lessons. Students apply their understanding of operating with directed numbers to solve problems in context. The first problem involves a fish tank that is being filled and drained. The second problem deals with historic voyages in a bathyscaphe (deep-sea submarine) and a high-altitude hot air balloon.

The activities in this lesson involve more reading than most lessons. Be prepared to support students with unfamiliar words.

Addressing

- Solve multi-step real-life and mathematical problems posed with positive and negative rational numbers in any form (whole numbers, fractions, and decimals), using tools strategically. Apply properties of operations to calculate with numbers in any form; convert between forms as appropriate; and assess the reasonableness of answers using mental computation and estimation strategies.
- Solve real-world and mathematical problems involving the four operations with rational numbers. Calculations with rational numbers extend the rules for manipulating fractions to complex fractions.
- Recognise and represent proportional relationships between quantities.

Instructional Routines

- Clarify, Critique, Correct
- Three Reads

- Discussion Supports
- Think Pair Share

Student Learning Goals

Let's apply what we know about directed numbers.

12.1 Grapes per Minute

Warm Up: 5 minutes

The purpose of this warm-up is to review "per" language.

Launch

Arrange students in groups of 2. Give students 1 minute of quiet work time followed by 1 minute of partner discussion, then follow with whole-class discussion.

Student Task Statement

1. If you eat 5 grapes per minute for 8 minutes, how many grapes will you eat?
2. If you hear 9 new songs per day for 3 days, how many new songs will you hear?
3. If you run 15 laps per practice, how many practices will it take you to run 30 laps?

Student Response

1. 40
2. 27
3. 2

Activity Synthesis

For each question, ask a student to share their response and reasoning. Resolve any disagreements that come up. Remind students that whenever we see the word "per", that means "for every 1".

12.2 Water Level in the Aquarium

10 minutes

This activity builds students understanding of how negative rates can be used to model directed change. Students use their knowledge of dividing and multiplying negative numbers to answer questions involving rates. They are not expected to express these as relationships of the form $y = kx$ in this activity, though some students might. In the second question students will also need to convert between different rates. Pay close attention to students conversion between units in the last question as the rates are over different times. Identify those who convert to the same units (either minutes or hours).

Instructional Routines

- Three Reads
- Discussion Supports

Launch

For each question, ask students to read the whole prompt individually before they start working on it. Remind students that they need to justify their answers. At the end of the activity put the students into groups of two. Use *Discussion Supports* to help students create context for this activity by providing or guiding students in creating visual diagrams that illustrate what is happening with the aquariums.

Representation: Access for Perception. Read the statements aloud. Students who both listen to and read the information will benefit from extra processing time. If students are unsure where to begin, suggest that they draw a diagram to help organise the information provided.
Supports accessibility for: Language Reading: Three Reads. Use this routine with the first problem to support students' reading comprehension. In the first read, students read the situation with the goal of understanding what the situation is about (e.g., an aquarium has a system to maintain the correct water level; too much water and it will overflow; too little water and the fish will get sick). If needed, discuss the meaning of unfamiliar terms at this time, or display a diagram to clarify the context. Use the second read to identify the important quantities by asking students what can be counted or measured, without focusing on specific values. Listen for, and amplify, the important quantities that vary in relation to each other: amount of water in the aquarium, in litres; amount of time spent filling or draining water, in minutes. After the third read, ask students to brainstorm possible strategies to answer the questions. This helps students connect the language in the word problems with the reasoning needed to solve the problems.
Design Principle(s): Support sense-making

Student Task Statement

1. A large aquarium should contain 10 000 litres of water when it is filled correctly. It will overflow if it gets up to 12 000 litres. The fish will get sick if it gets down to 4 000 litres. The aquarium has an automatic system to help keep the correct water level. If the water level is too low, a tap fills it. If the water level is too high, a drain opens.

   One day, the system stops working correctly. The tap starts to fill the aquarium at a rate of 30 litres per minute, and the drain opens at the same time, draining the water at a rate of 20 litres per minute.

   a. Is the water level rising or falling? How do you know?

   b. How long will it take until the tank starts overflowing or the fish get sick?

2. A different aquarium should contain 15 000 litres of water when filled correctly. It will overflow if it gets to 17 600 litres.

One day there is an accident and the tank cracks in 4 places. Water flows out of each crack at a rate of $\frac{1}{2}$ litre per hour. An emergency pump can re-fill the tank at a rate of 2 litres per minute. How many minutes must the pump run to replace the water lost each hour?

Student Response

1. Represent the filling as 30 litres per minute and the drain as -20 litres per minute.

    a. This means that the water level is rising at $30 + (-20)$ or 10 litres per minute.

    b. The tank needs an extra 2 000 litres before it overflows. At 10 litres per minute this takes $2000 \div 10$ or 200 minutes.

2. The tank has 4 cracks, which leak at 0.5 litres per hour. This is a total of -2 litres per hour. The emergency pump is 2 litres per minute, so must run for one minute every hour to compensate.

Activity Synthesis

Ask students to compare their simplifying assumptions and solution methods. In a whole group discussion bring out the assumptions students have made. Compare the different rates in the answers to the last question.

12.3 Up and Down with the Piccards

15 minutes

This activity builds on students' previous work with proportional relationships, as well as their understanding of multiplying and dividing directed numbers, to model different historical scenarios involving ascent and descent, and students must explain their reasoning. While equations of the form $y = kx$ are not technically proportional relationships if k is negative, students can still work with these equations. Identify students who convert between seconds and hours.

Instructional Routines

- Clarify, Critique, Correct
- Discussion Supports
- Think Pair Share

Launch

Arrange students in groups of 2. Introduce the activity by asking students where they think the deepest part of the ocean is. It may be helpful to display a map showing the location of the Challenger Deep, but this is not required. Explain that Jacques Piccard had to design a specific type of submarine to make such a deep descent. Use *Discussion Supports* to help students create context for this activity. Provide pictures of Jacques Piccard with a submersible and Auguste Piccard with a hot air balloon. Guide students in creating visual diagrams that illustrate what is happening with the trench, ocean surface, and balloon. Remind students that they need to explain their reasoning. Provide for a quiet work time followed by partner and whole-class discussion.

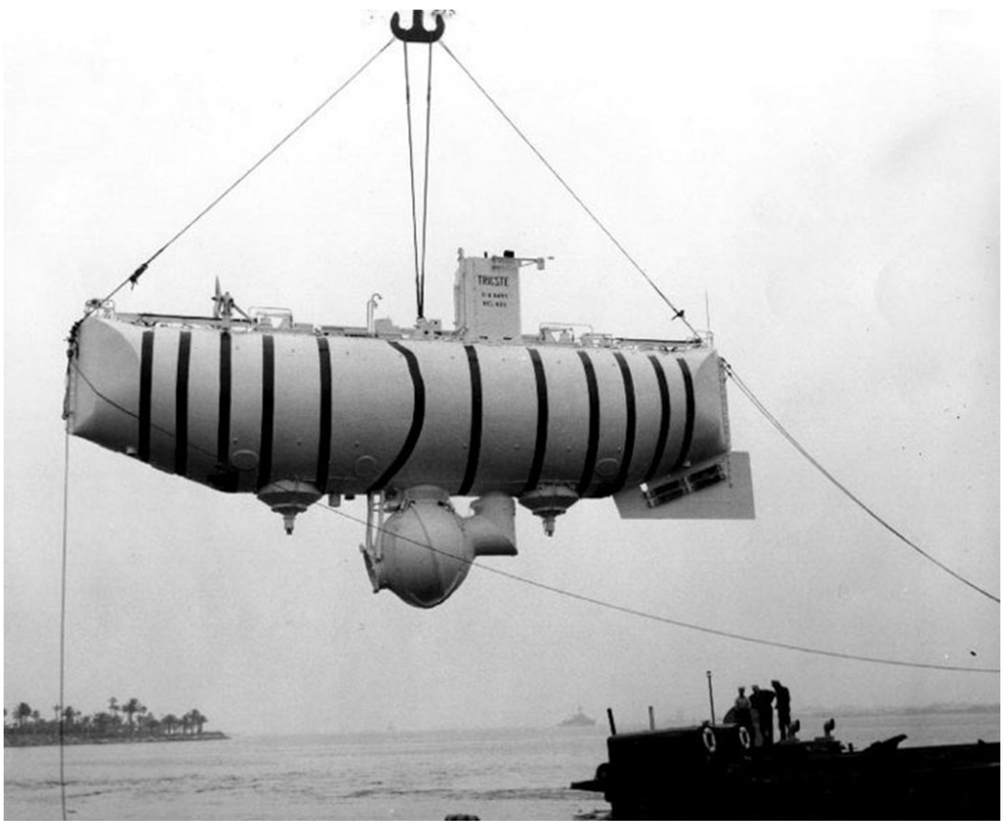
Representation: Internalise Comprehension. Provide appropriate reading accommodations and supports to ensure students have access to written directions, word problems and other text-based content.
Supports accessibility for: Language; Conceptual processing

Anticipated Misconceptions

Some students may be confused by the correct answer to the last question, thinking that $51\,683 - 52\,940 = -1\,257$means that Auguste landed his balloon below sea level. Explain that Auguste launched his balloon from a mountain, to help him reach as high of an altitude as possible. We have chosen to use zero to represent this starting point, instead of sea level, for this part of the activity. Therefore, a vertical position of -1 257 feet means that Auguste landed below his starting point, but not below sea level.

Student Task Statement

1. Challenger Deep is the deepest known point in the ocean, at 35 814 feet below sea level. In 1960, Jacques Piccard and Don Walsh rode down in the Trieste and became the first people to visit the Challenger Deep.



a. If sea level is represented by 0 feet, explain how you can represent the depth of a submarine descending from sea level to the bottom of Challenger Deep.

b. Trieste's descent was a change in depth of -3 feet per second. We can use the relationship $y = \text{-}3x$ to model this, where y is the depth (in feet) and x is the time (in seconds). Using this model, how much time would the Trieste take to reach the bottom?

c. It took the Trieste 3 hours to ascend back to sea level. This can be modelled by a different relationship $y = kx$. What is the value of k in this situation?

2. The design of the Trieste was based on the design of a hot air balloon built by Auguste Piccard, Jacques's father. In 1932, Auguste rode in his hot-air balloon up to a record-breaking height.

   a. Auguste's ascent took 7 hours and went up 51 683 feet. Write a relationship $y = kx$ to represent his ascent from his starting location.

   b. Auguste's descent took 3 hours and went down 52 940 feet. Write another relationship to represent his descent.

   c. Did Auguste Piccard end up at a greater or lesser altitude than his starting point? How much higher or lower?

Student Response

1.

   a. We can use negative numbers to represent how many feet below sea level a submarine is.

   b. It would take 11 938 seconds to descend to the sea floor, or about 3 hours and 19 minutes. Substituting the depth of -35 814 in for y gives the equation $\text{-}35\,814 = \text{-}3x$. Solving the equation gives 11 938 for x, because $\text{-}35\,814 \div (\text{-}3) = 11\,938$.

   c. $k = \frac{35\,814}{10\,800}$; It took 10 800 seconds to go up 35 814 feet, because $3 \times 60 \times 60 = 10\,800$. (This is a vertical change of approximately 3.32 feet per second.)

2.

   a. The relationship is $y = \frac{51\,683}{25\,200}x$. It took 25 200 seconds to go up 51 683 feet, because $7 \times 60 \times 60 = 25\,200$. (This is a vertical change of approximately 2.05 feet per second.)

   b. The relationship is $y = \frac{\text{-}52\,940}{10\,800}x$, because it took 10 800 seconds to go down 52 940 feet. (This is a vertical change of approximately -4.9 feet per second.)

c. He ended up lower because $51\,683 - 52\,940 = -1\,257$, so $1\,257$ feet lower than his starting point.

Are You Ready for More?

During which part of either trip was a Piccard changing vertical position the fastest? Explain your reasoning.

- Jacques's descent
- Jacques's ascent
- Auguste's ascent
- Auguste's descent

Student Response

Auguste's descent was the fastest.

Activity Synthesis

First, have students compare their solutions with a partner and describe what is the same and what is different. This will help students be prepared to explain their reasoning to the whole class.

Next, select students to share with the class. Highlight solutions that correctly operate with negatives, those who convert between seconds and hours, and those that state their assumptions clearly. Help students make sense of each equation by asking questions such as:

- After 1 second, by how much have they changed vertical position? After 10 seconds? After 100 seconds?
- How can you tell from the equation whether they are going up or down?
- How can you tell from the equation the total distance or total time of their ascent or descent?
- What does 0 represent in this situation?

The most important thing for students to get out of this activity is how different operations with directed numbers were helpful for representing the situation and solving the problems.

Writing, Representing: Clarify, Critique, Correct. To begin the whole-class discussion, display the following incorrect statements for all to see: “Since they are under water, submarines have negative ascent and descent, so the k will always be negative." and "A hot air balloon will always have a positive ascent and descent because it is in the air, so the k will always be negative." Give students 1–2 minutes of quiet think time to identify the errors, and to

critique the reasoning. Invite students to share their thinking with a partner, and to work together to correct one of the statements. Listen for the ways students interpret each statement, paying attention to the connections they make between vertical distances, and how changes in vertical position are represented. For each situation, select 1–2 groups to share their revised statements with the class. This will help clarify language and understanding about how ascent and descent can be represented by an equation.
Design Principle(s): Optimise output (for explanation); Maximise meta-awareness

Lesson Synthesis

Key takeaways:

- Recall we can represent speed with direction (velocity) using directed numbers. We can do this with vertical movement (in fact with any rate).
- The convention is that up is the positive direction and down is the negative direction.

Discussion questions:

- What other rates have you encountered where it makes sense to have positive and negative values?

12.4 Submarines

Cool Down: 5 minutes

Student Task Statement

1. A submarine is descending to examine the seafloor 2 100 feet below the surface. It takes the submarine 2 hours to make this descent. Write an equation to represent the relationship between the submarine's height above sea level and time.
2. Another submarine's descent can be represented as $y = \text{-}240x$, where y is the height above sea level and x is time in hours. How long will it take this submarine to make the descent?

Student Response

1. $y = \text{-}1\,050x$ where y is the height above sea level and x is the hour
2. 8.75 hours

Student Lesson Summary

We saw earlier that we can represent speed with direction using directed numbers. Speed with direction is called *velocity*. Positive velocities always represent movement in the opposite direction from negative velocities.

We can do this with vertical movement: moving up can be represented with positive numbers, and moving down with negative numbers. The magnitude tells you how fast, and the sign tells you which direction. (We could actually do it the other way around if we wanted to, but usually we make up positive and down negative.)

Lesson 12 Practice Problems

1. **Problem 1 Statement**

   Describe a situation where each of the following quantities might be useful.

   a. -20 gallons per hour

   b. -10 feet per minute

   c. -0.1 kilograms per second

   Solution

   Answers vary. Sample responses:

   a. Water leaking out of a tank

   b. An airplane descending

   c. Gravel being emptied out of a truck

2. **Problem 2 Statement**

   A submarine is only allowed to change its depth by rising toward the surface in 60-metre stages. It starts off at -340 metres.

   a. At what depth is it after:

      i. 1 stage

      ii. 2 stages

      iii. 4 stages

   b. How many stages will it take to return to the surface?

   Solution

      i. -280 m; $-340 + 60 = -280$

      ii. -220 m; $-340 + 120 = -220$

      iii. -100m; $-340 + 240 = -100$

   a. 6 stages of change. $340 \div 60 = 5.7$ so the submarine would need 6 stages

3. **Problem 3 Statement**

Some boats were travelling up and down a river. A satellite recorded the movements of several boats.

a. A motor boat travelled -3.4 miles per hour for 0.75 hours. How far did it go?

b. A tugboat travelled -1.5 miles in 0.3 hours. What was its velocity?

c. What do you think that negative distances and velocities could mean in this situation?

Solution

a. -2.55 miles; $-3.4 \times 0.75 = -2.55$

b. -5 miles per hour; $-1.5 \div 0.3 = -5$

c. Someone had to choose one direction to be positive and the other to be negative. Positive distances could mean distances in the positive direction and negative distances mean distances in the other direction. Positive velocities could mean it was moving in the positive direction and negative velocities mean it was moving in the negative direction.

4. **Problem 4 Statement**

a. A cookie recipe uses 3 cups of flour to make 15 cookies. How many cookies can you make with this recipe with 4 cups of flour? (Assume you have enough of the other ingredients.)

b. A teacher uses 36 centimetres of tape to hang up 9 student projects. At that rate, how much tape would the teacher need to hang up 10 student projects?

Solution

a. 20 cookies; based on the information one can determine that 1 cup of flour makes 5 cookies, so 4 cups of flour will make 20 cookies.

b. 40 centimetres; based on the information one can determine that 4 centimetres of tape will hang up 1 student project, so the teacher would need 40 centimetres of tape to hang 10 student projects.

5. **Problem 5 Statement**

Evaluate each expression. When the answer is not a whole number, write your answer as a fraction.

a. -4×-6

b. $-24 \times \frac{-7}{6}$

c. $4 \div -6$

d. $\frac{4}{3} \div -24$

Solution

a. 24

b. 28

c. $\frac{-2}{3}$ (or equivalent)

d. $\frac{-1}{18}$ (or equivalent)