Lesson 15: The volume of a cone

Goals

- Calculate the volume of a cone and cylinder given the height and radius, and explain (orally) the solution method.
- Compare the volumes of a cone and a cylinder with the same base and height, and explain (orally and in writing) the relationship between the volumes.

Learning Targets

- I can find the volume of a cone in mathematical and real-world situations.
- I know the formula for the volume of a cone.

Lesson Narrative

In this lesson students start working with cones, and learn that the volume of a cone is $\frac{1}{3}$ the volume of a cylinder with a congruent base and the same height. First, students learn a method for quickly sketching a cone, and the meaning of the *radius* and *height* of a cone. Then they watch a video (or if possible, a live demonstration) showing that it takes three cones of water to fill a cylinder with the same radius and height. At this point, it is taken as a mysterious and beautiful fact that the volume of a cone is one third the volume of the associated cylinder. A proof of this fact requires mathematics beyond this level.

Students write the volume of a cone given a specific volume of a cylinder with the same base and height, and vice versa. Then they use the formula for the volume of a cylinder learned in previous lessons to write the general formula $V = \frac{1}{3}\pi r^2 h$ for the volume, V, of a cone in terms of its height, h, and radius, r. Finally, students practise calculating the volumes of some cones. There are opportunities for further practice in the next lesson.

Addressing

- Know the formulae for the volumes of cones, cylinders, and spheres and use them to solve real-world and mathematical problems.

Instructional Routines

- Clarify, Critique, Correct
- Discussion Supports
- Notice and Wonder
- Think Pair Share

Required Preparation

For the Which Has a Larger Volume activity, it is suggested that students have access to geometric solids.

During the From Cylinders to Cones activity, students will need to view a video. Alternatively, do a demonstration with a cone that could be filled with water and poured into a cylinder.

Student Learning Goals

Let’s explore cones and their volumes.

15.1 Which Has a Larger Volume?

Warm Up: 5 minutes

The purpose of this activity is for students to think about how the volume of a cone might relate to the volume of a cylinder with the same base and height. Additionally, students learn one method for sketching a cone. In this activity, just elicit students’ best guess about how many cone-contents would fit into the cylinder (or, what fraction of the cylinder’s volume is the cone’s volume). In the next activity, they will watch a demonstration that verifies the actual amount.

Instructional Routines

- Think Pair Share

Launch

If you have access to appropriate geometric solids that include a cylinder and a cone with congruent bases and equal heights, consider showing these to students, even passing them around for students to hold if time permits.

Arrange students in groups of 2. Give students 2–3 minutes of quiet work time, followed by time to discuss fractional amount with partner. Follow with a whole-class discussion.

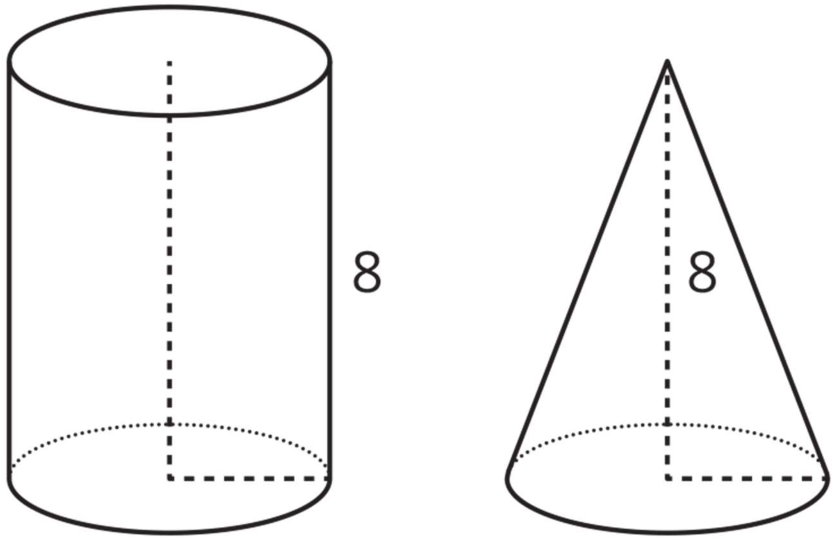
Anticipated Misconceptions

If students think the two shapes will have the same volume, ask them to imagine dropping the cone into the cylinder and having extra space around the cone and still inside the cylinder.

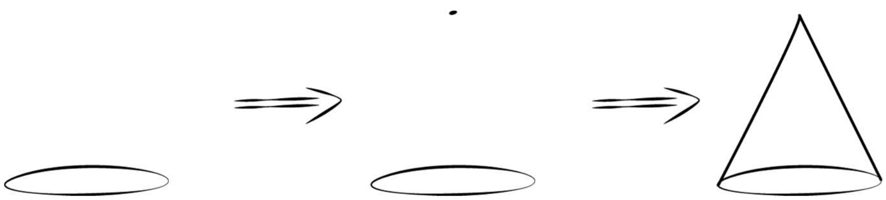
Student Task Statement

The cone and cylinder have the same height, and the radii of their bases are equal.

1. Which figure has a larger volume?
2. Do you think the volume of the smaller one is more or less than $\frac{1}{2}$ the volume of the larger one? Explain your reasoning.
3. Sketch two different sized cones. The oval doesn’t have to be on the bottom! For each drawing, label the cone’s radius with r and height with h.



Here is a method for quickly sketching a cone:



- Draw an oval.
- Draw a point centred above the oval.
- Connect the edges of the oval to the point.
- Which parts of your drawing would be hidden behind the object? Make these parts dashed lines.

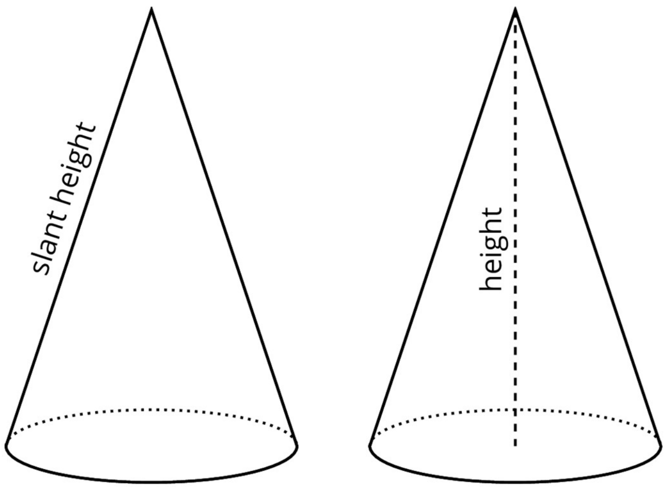
Student Response

1. The cylinder has a larger volume.
2. Answers vary.
3. Answers vary.

Activity Synthesis

Invite students to share their answers to the first two questions. The next activity includes a video that shows that it takes 3 cones to fill a cylinder that has the same base and height as the cone, so it is not necessary that students come to an agreement about the second question, just solicit student's best guesses, and tell them that we will find out the actual fractional amount in the next activity.

End the discussion by selecting 2–3 students to share their sketches. Display these for all to see and compare the different heights and radii. If no student draws a perpendicular height or slant height, display the image shown here for all to see and remind students that in earlier units we learned that height creates a right angle with something in the shape. In the case of the cones, the height is perpendicular to the circular base.



15.2 From Cylinders to Cones

20 minutes

In this activity, students use the relationship that the volume of a cone is $\frac{1}{3}$ of the volume of a cylinder to calculate the volume of various cones. Students start by watching a video (or demonstration) that shows that it takes the contents of 3 cones to fill the cylinder when they have congruent bases and equal heights. Students use this information to calculate the volume of various cones and cylinders. For the last question, identify students who:

- write the equation as $\frac{1}{3}V$ (or $V \div 3$), where V represents the volume of a cylinder with the same base and height as the cone.
- write the equation in terms of r and h ($V = \frac{1}{3}\pi r^2 h$).

Instructional Routines

- Discussion Supports
- Notice and Wonder

Launch

Video 'How Many Cones Does it Take to Fill a Cylinder with the Same Base and Height?' available here: https://player.vimeo.com/video/309581286.

Either conduct a demonstration or show the video and tell students to write down anything they notice or wonder while watching. Pause for a whole-class discussion. Record what students noticed and wondered for all to see. Ensure that students notice that it takes the contents of 3 cones to fill the cylinder, or alternatively, that the volume of the cone is $\frac{1}{3}$ the volume of the cylinder. Then, set students to work on the questions in the task, followed by a whole-class discussion.

Speaking: Discussion Supports. As students describe what they notice and wonder about from the video, revoice student ideas to demonstrate mathematical language use while incorporating gestures and referring to the context in the video. This will help students to produce and make sense of the language needed to communicate their own ideas about the relationship between the volume of cones and cylinders.
Design Principle(s): Support sense-making; Optimise output (for explanation)

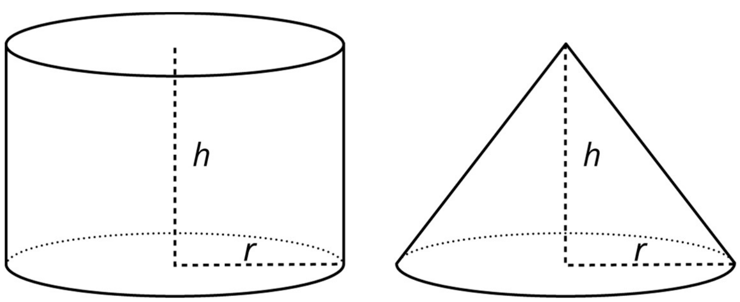
Student Task Statement

A cone and cylinder have the same height and their bases are congruent circles.

1. If the volume of the cylinder is 90 cm^3, what is the volume of the cone?

2. If the volume of the cone is 120 cm^3, what is the volume of the cylinder?

3. If the volume of the cylinder is $V = \pi r^2 h$, what is the volume of the cone? Either write an expression for the cone or explain the relationship in words.



Student Response

1. 30 cm^3. The volume of a cone is $\frac{1}{3}$ the volume of a cylinder and $\frac{1}{3} \times 90$.

2. 360 cm^3. The volume of the cylinder is 3 times larger than the volume of a cone and 3×120.

3. Answers vary. Sample responses: $V = \frac{1}{3}\pi r^2 h$ (or equivalent) or “the volume of the cone is $\frac{1}{3}$ that of the volume of the cylinder.”

Activity Synthesis

Select previously identified students to share the volume equation they wrote for the last question. Display examples for all to see and ask “Are these equations the same? How can you know for sure?” (The calculated volume is the same when you use both equations.)

If no student suggests it, connect $\frac{1}{3}V$, where V represents the volume of a cylinder with the same base and height as the cone, to the volume of the cone, $\frac{1}{3}\pi r^2 h$. Reinforce that these are equivalent expressions.

Add the formula $V = \frac{1}{3}\pi r^2 h$ and a diagram of a cone to your classroom displays of the formulae being developed in this unit.

15.3 Calculate That Cone

10 minutes

The purpose of this activity is for students to calculate the volume of cones given their height and radius. Students are given a cylinder with the same height and radius and use the volume relationship they learned in the previous activity to calculate the volume of the cone. They then calculate the volume of a cone given a height and radius using the newly learned formula for volume of a cone. For the last problem, an image is not provided to give students the opportunity to sketch one if they need it.

Instructional Routines

- Clarify, Critique, Correct

Launch

Give students 3–5 minutes of quiet work time followed by a whole-class discussion.

Anticipated Misconceptions

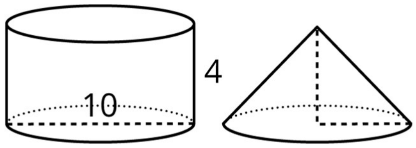
In the first problem, students might use 10 as the radius length. Ask students what the length of 10 in the picture is called. Ask students to recall the formula for the volume of a cylinder.

For students who are not sure where to begin the last problem since it does not have an image, encourage them to sketch and label their own.

Student Task Statement

1. Here is a cylinder and cone that have the same height and the same base area.

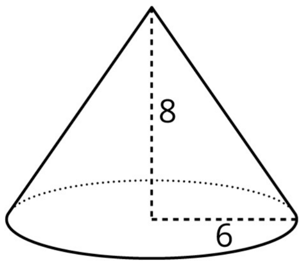
   What is the volume of each shape? Express your answers in terms of π.



2. Here is a cone.

   a. What is the area of the base? Express your answer in terms of π.

   b. What is the volume of the cone? Express your answer in terms of π.



3. A cone-shaped popcorn cup has a radius of 5 centimetres and a height of 9 centimetres. How many cubic centimetres of popcorn can the cup hold? Use 3.14 as an approximation for π, and give a numerical answer.

Student Response

1. Cylinder: 100π; cone: $\frac{100}{3}\pi$. To calculate the volume of the cylinder, find the area of the base and multiply it by the height of the cylinder. The area of the base is πr^2, and the height is 4 ($\pi \times 5^2 \times 4 = 100\pi$). The volume of the cone is $\frac{1}{3}$ of the cylinder's volume, which is $\frac{100}{3}\pi$.

2.
    a. 36π square units because $A = \pi \times 6^2$.

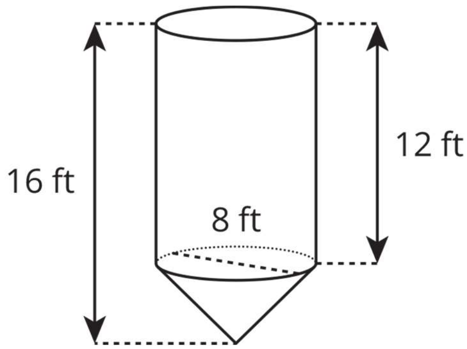
    b. 96π cubic units because the volume is $\frac{1}{3}$ of the area of the base multiplied by the height of the cone ($\frac{1}{3} \times 36\pi \times 8 = 96\pi$).

3. 235.5 cm^3 because with a radius of 5 cm and a height of 9 cm, the volume is calculated with the equation $V = \frac{1}{3} \times 3.14 \times 5^2 \times 9$.

Are You Ready for More?

A grain silo has a cone shaped spout on the bottom in order to regulate the flow of grain out of the silo. The diameter of the silo is 8 feet. The height of the cylindrical part of the silo above the cone spout is 12 feet while the height of the entire silo is 16 feet.

How many cubic feet of grain are held in the cone spout of the silo? How many cubic feet of grain can the entire silo hold?



Student Response

The entire grain silo holds $\frac{640}{3}\pi$ cubic feet of grain.

The cone holds $\frac{64}{3}\pi$ cubic feet of grain. Since the radius is 4 feet and the height of the cone is also 4 feet $(16 - 12)$, the volume is $\frac{1}{3}\pi 4^2 \times 4$. Calculate the volume of the cylinder $(\pi 4^2 \times 12)$, and add it to the volume of the cone to get the volume of the entire silo.

Activity Synthesis

For the first problem, Invite students to explain how they calculated the volume of both figures and have them share the different strategies they used. If not mentioned by students bring up these strategies:

- Calculate the volume of the cylinder, then divide volume of cylinder by 3 to get the volume of the cone.
- Calculate the volume of the cylinder, then multiply volume of cylinder by $\frac{1}{3}$ to get the volume of the cone.
- Calculate the volume of the cone, then multiply volume of cone by 3 to get the volume of the cylinder.

For the third problem, ask students to share any sketches they came up with to help them calculate the answer. Explain to students that sometimes we encounter problems that don't have a visual example and only a written description. By using sketches to help to visualise what is being described in a problem, we can better understand what is being asked.

Engagement: Develop Effort and Persistence. Break the class into small discussion groups and then invite a representative from each group to report back to the whole class. All group members should be prepared to share if invited.
Supports accessibility for: Language; Social-emotional skills; Attention Writing: Clarify, Critique, Correct. Present an incorrect response to the first question that reflects a possible misunderstanding from the class. For example, "The volume is 4000π because $\pi \times 4 \times 10^2$." Prompt students to critique the reasoning (e.g., ask, "Do you agree with the author's reasoning? Why or why not?") and then write feedback to the author that identifies the

misconception and how to improve on his/her work. Listen for students who tie their feedback to the difference between the radius and diameter and use the academic vocabulary (e.g., height, radius, diameter, cylinder, cone, volume, etc.). This will help students evaluate, and improve on, the written mathematical arguments of others and highlight the distinction between and the importance of radii when calculating volume of cylinders and cones.
Design Principle(s): Maximise meta-awareness

Lesson Synthesis

Have students summarise the highlights of the lesson by asking:

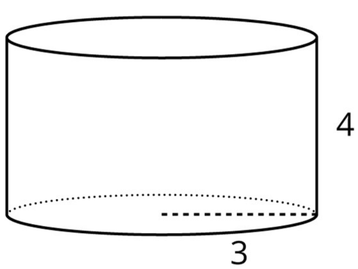
- "What is the relationship between the volume of a cylinder and the volume of a cone?" (The volume of a cone is $\frac{1}{3}$ of the volume of a cylinder or the volume of the cylinder is 3 times the volume of the cone.)
- "If we know the volume of a cone, how do we calculate the volume of a cylinder that has the same height and base area?" (We can multiply the volume of the cone by 3.)
- "If we know the volume of a cylinder, how do we calculate the volume of a cone that has the same height and base area?" (We can multiply the volume of the cylinder by $\frac{1}{3}$.)
- "If a cylinder and a cone have the same base, how tall does the cone have to be relative to the cylinder so that they both have the same volume?" (The cone needs to have a height 3 times the height of the cylinder for the two shapes to have the same volume.)

15.4 Calculate Volumes of Two Figures

Cool Down: 5 minutes

Student Task Statement

A cone with the same base but a height 3 times taller than the given cylinder exists. What is the volume of each figure? Express your answers in terms of π.



Student Response

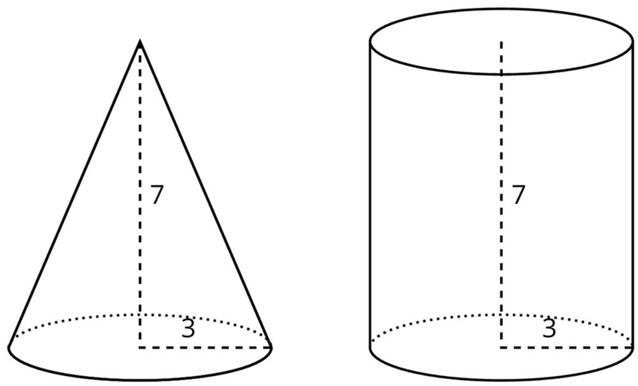
cylinder: 36π cubic units because $\pi \times 3^2 \times 4 = 36\pi$

cone: 36π cubic units because $\frac{1}{3}\pi \times 3^2 \times 12 = 36\pi$

Student Lesson Summary

If a cone and a cylinder have the same base and the same height, then the volume of the cone is $\frac{1}{3}$ of the volume of the cylinder. For example, the cylinder and cone shown here both have a base with radius 3 feet and a height of 7 feet.

The cylinder has a volume of 63π cubic feet since $\pi \times 3^2 \times 7 = 63\pi$. The cone has a volume that is $\frac{1}{3}$ of that, or 21π cubic feet.



If the radius for both is r and the height for both is h, then the volume of the cylinder is $\pi r^2 h$. That means that the volume, V, of the cone is $V = \frac{1}{3}\pi r^2 h$

Lesson 15 Practice Problems

1. **Problem 1 Statement**

   A cylinder and cone have the same height and radius. The height of each is 5 cm, and the radius is 2 cm. Calculate the volume of the cylinder and the cone.

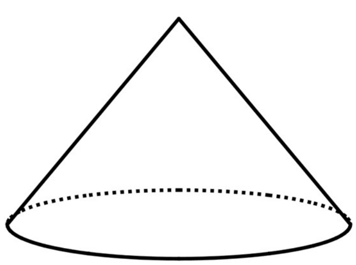
   Solution

   Cylinder: 20π cm^3, cone: $\frac{20}{3}\pi$ cm^3

2. **Problem 2 Statement**

   The volume of this cone is 36π cubic units.

   What is the volume of a cylinder that has the same base area and the same height?



Solution

108π, about 339 cubic units (The volume of the cylinder is exactly three times the volume of the corresponding cone.)

3. **Problem 3 Statement**

A cylinder has a diameter of 6 cm and a volume of 36π cm^3.

a. Sketch the cylinder.

b. Find its height and radius in centimetres.

c. Label your sketch with the cylinder's height and radius.

Solution

Answers vary. The radius of the cylinder is 3 cm and the height of the cylinder is 4 cm since $\frac{36}{3^2} = 4$.

4. **Problem 4 Statement**

Lin wants to get some custom T-shirts printed for her basketball team. Shirts cost £10 each if you order 10 or fewer shirts and £9 each if you order 11 or more shirts.

a. Make a graph that shows the total cost of buying shirts, for 0 through 15 shirts.

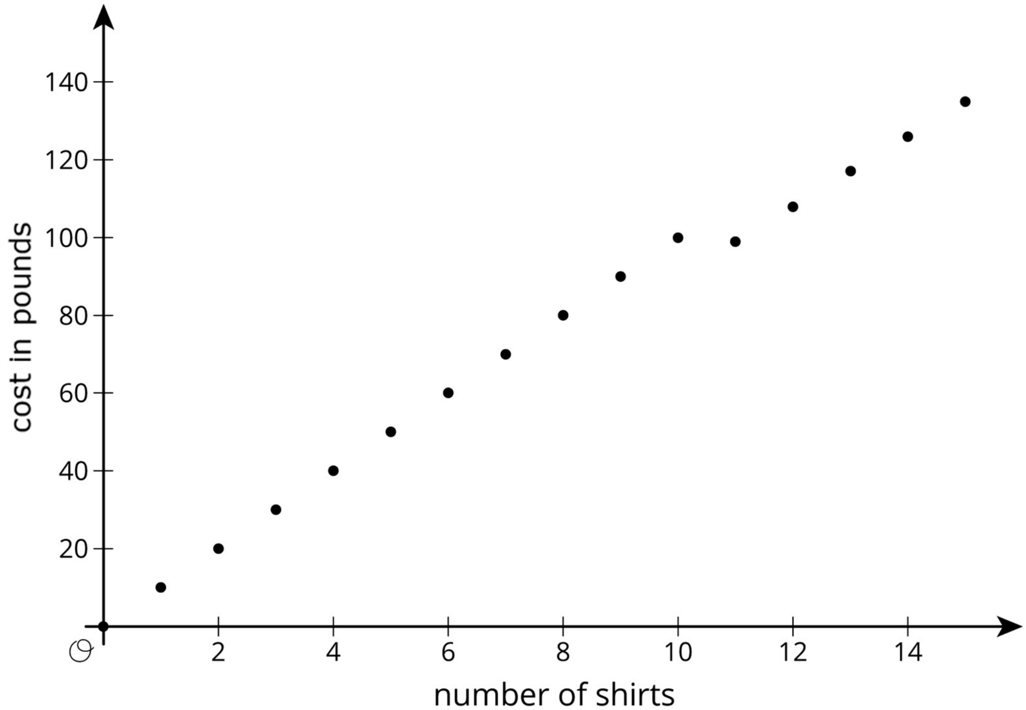
b. There are 10 people on the team. Do they save money if they buy an extra shirt? Explain your reasoning.

c. What is the gradient of the graph between 0 and 10? What does it mean in the story?

d. What is the gradient of the graph between 11 and 15? What does it mean in the story?

Solution

a.



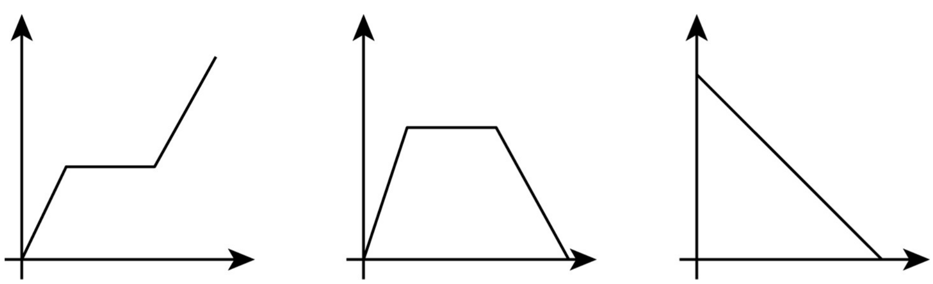
b. Yes, 11 shirts cost £99, and 10 shirts cost £100. Even if they split the cost of the extra shirt, they still save £1 altogether, or £0.10 apiece. If they can find someone to buy the extra shirt, they save £1 each.

c. 10 pounds per shirt: the price per shirt when you buy 10 or fewer

d. 9 pounds per shirt: the price per shirt when you buy 11 or more

5. **Problem 5 Statement**

In the following graphs, the horizontal axis represents time and the vertical axis represents distance from school. Write a possible story for each graph.



Solution

Answers vary. Sample response:

– Left graph: A student leaves school and walks to a friend's house that is halfway between school and their house. After spending some time at a friend's house, the student continues walking home.

- Centre graph: An athlete leaves school to go home for a while, then returns to school for a game later in the evening.
- Right graph: A parent starts from their workplace and drives directly to school to pick up their daughter.

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