Lesson 8: How many solutions?

Goals

- Describe (orally) a linear equation as having “one solution”, “no solutions”, or “an infinite number of solutions”, and solve equations in one variable with one solution.
- Describe (orally) features of linear equations with one solution, no solution, or an infinite number of solutions.

Learning Targets

- I can solve equations with different numbers of solutions.

Lesson Narrative

In the previous lesson, students learned that sometimes an equation has one solution, sometimes no solution, and sometimes infinitely many solutions. The purpose of this lesson is to help students identify structural features of an equation that tell them which of these outcomes will occur when they solve it. They also learn to stop solving an equation when they have reached a point where it is clear which of the outcomes will occur, for example when they reach an equation like $6x + 2 = 6x + 5$ (no solution) or $6x + 2 = 6x + 2$ (infinitely many solutions).

Addressing

- Give examples of linear equations in one variable with one solution, infinitely many solutions, or no solutions. Show which of these possibilities is the case by successively transforming the given equation into simpler forms, until an equivalent equation of the form $x = a$, $a = a$, or $a = b$ results (where a and b are different numbers).

Instructional Routines

- Stronger and Clearer Each Time
- Discussion Supports

Required Materials

Pre-printed slips, cut from copies of the blackline master

Thinking About Solutions Some More A $7(x-5)=x+13$	Thinking About Solutions Some More B $-6x=-5(x-1)-x$
Thinking About Solutions Some More C $-4(x-2)=-2\left(x-\frac{17}{2}\right)$	Thinking About Solutions Some More D $3-4x+5=2(8-2x)$
Thinking About Solutions Some More E $2x+3=3+2x$	Thinking About Solutions Some More F $2x+3=2x+5$
Thinking About Solutions Some More G $3x+9=2.5x+14$	Thinking About Solutions Some More H $7(x-4)=4x+5$
Thinking About Solutions Some More I $5x-20=-7x-20$	Thinking About Solutions Some More J $3(2x+1)-4x=2x+3$

Required Preparation

Make 1 copy of the Make Use of Structure blackline master for every 3 students, and cut them up ahead of time.

Student Learning Goals

Let’s solve equations with different numbers of solutions.

8.1 Matching Solutions

Warm Up: 5 minutes

Students extend their understanding from the previous lessons to recognise the structure of a linear equation for all possible types of solutions: one solution, no solution, or infinitely many solutions. Students are still using language such as "true for one value of x," "always true" or "true for any value of x," and "never true." Students should be able to articulate that this depends both on the **coefficient** of the variable as well as the constant term on each side of the equation.

Launch

Give students 2–3 minutes of quiet think time followed by a whole-class discussion.

Student Task Statement

Consider the unfinished equation $12(x-3)+18=$ _________. Match the following expressions with the number of solutions the equation would have with that expression on the right hand side.

1. $6(2x-3)$
2. $4(3x-3)$
3. $4(2x-3)$

- one solution
- no solutions
- infinitely many solutions

Student Response

1. The equation $12(x-3)+18=6(2x-3)$ has infinitely many solutions, or is true for any value of x.
2. The equation $12(x-3)+18=4(3x-3)$ has no solution.
3. The equation $12(x-3)+18=4(2x-3)$ has one solution, $x=\frac{3}{2}$.

Activity Synthesis

In order to highlight the structure of these equations, ask students:

- "What do you notice about equations with no solution?" (These equations have equal or equivalent coefficients for the variable, but unequal values for the constants on each side of the equation.)

- “What do you notice about equations that have infinitely many solutions?” (These equations have equivalent expressions on each side of the equation, so the coefficients are equal and the constants are equal or equivalent on each side.)
- “What do you notice about equations that have exactly one solution?” (These equations have different values for the coefficients on each side of the equation and it doesn’t matter what the constant term says.)

Display the equation $x = 12$ for all to see. Ask students how this fits with their explanations. (We can see that there is one solution. Another way to think of this is that the coefficient of x is 1 on the left side of the equation, and the coefficient of x is 0 on the right side of the equation. So the coefficients of x are different, just like the explanation.)

8.2 Thinking About Solutions Some More

25 minutes

In this activity students solve a variety of equation types; both in form and number of solutions. After solving the 10 equations, groups sort them into categories of their choosing. The goal of this activity is to encourage students to look at the structure of equations before solving and to build fluency solving complex equations. For example, students who notice that equation D, $3 - 4x + 5 = 2(8 - 2x)$, has the same number of xs on each side but a different constant know that there are no values of x that make the equation true. Similarly, equation J has the same number of xs on each side and the same constants on each side, meaning that all values of x make the equation true. These up-front observations allow students to avoid spending time working out the steps to re-write the equation into a simpler form where the number of solutions to the equations is easier to see.

Instructional Routines

- Discussion Supports

Launch

Arrange students in groups of 3. Distribute 10 pre-cut slips from the blackline master to each group. After groups have solved and sorted their equations, consider having groups switch to examine another group’s categories. Leave 3–4 minutes for a whole-class discussion.

Representation: Internalise Comprehension. Chunk this task into more manageable parts to differentiate the degree of difficulty or complexity by beginning with fewer cards. For example, give students a subset of the cards to start with and introduce the remaining cards once students have completed their initial sort. Some students may benefit from an additional set of cards with more accessible values to start with.
Supports accessibility for: Conceptual processing; Organisation

Student Task Statement

Your teacher will give you some cards.

1. With your partner, solve each equation.
2. Then, sort them into categories.
3. Describe the defining characteristics of those categories and be prepared to share your reasoning with the class.

Student Response

Answers vary. Possible solution:

- 5 equations of varying complexity, some including brackets and some not, that have a single integer solution.
  - A $7(x-5)=x+13$ ($x=8$)
  - C $-4(x-2)=-2\left(x-\frac{17}{2}\right)$ ($x=-4.5$)
  - G $3x+9=2.5x+14$ ($x=10$)
  - H $7(x-4)=4x+5$ ($x=11$)
  - I $5x-20=-7x-20$ ($x=0$)
- 3 equations with no solution.
  - F $2x+3=2x+5$
  - B $-6x=-5(x-1)-x$
  - D $3-4x+5=2(8-2x)$
- 2 equations with an infinite number of solutions.
  - E $2x+3=3+2x$
  - J $3(2x+1)-4x=2x+3$

Activity Synthesis

Select 2–3 groups to share one of their categories' defining characteristics and which equations they sorted into it. Given the previous activity, the categories "one solution, no solution, all (infinitely many) values for x are solutions" are likely. Introduce students to the language "infinite number of solutions" if it has not already come up in discussion.

During the discussion, it is likely that students will want to refer to specific parts of an expression. Encourage students to use the words "coefficient" and "variable." Define the word **constant term** as the term in an expression that doesn't change, or the term that does not have a variable part. For example, in the expression $2x - 3$, the 2 is the coefficient of x, the 3 is a constant, and x is the variable.

Representing, Speaking: Discussion Supports. To support the whole-class discussion, provide the following sentence frames when groups share: "We grouped these equations into the category ___ because____." and "Some characteristics of this category are ___." Invite students to press for details by asking clarifying questions (e.g., "What other features do those equations have in common?" and "Can you explain how to create a different equation that would fall into that category?"). Emphasise language students use to define parts of an expression such as "variable" and "coefficient," and use this opportunity to define the word "constant term." This will help students generalise the categories of equations that have one solution, no solution, or "infinitely many solutions."
Design Principle(s): Support sense-making

8.3 Make Use of Structure

Optional: 15 minutes

The purpose of this activity is so that students can compare the structure of equations that have no solution, one solution, and infinitely many solutions. This may be particularly useful for students needing more practice identifying which equations will have each of these types of solutions before attempting to solve.

Instructional Routines

- Stronger and Clearer Each Time

Launch

Arrange students in groups of 2. Give students 3–5 minutes of quiet think time followed by 2–3 minutes of partner discussion. Follow with a whole-class discussion.

Anticipated Misconceptions

On the second question, students may think that $x - 3 = 3 - x$ have the same coefficients of x. Recall that x has a coefficient of 1, while -x has a coefficient of -1.

Student Task Statement

For each equation, determine whether it has no solutions, exactly one solution, or is true for all values of x (and has infinitely many solutions). If an equation has one solution, solve to find the value of x that makes the statement true.

1.

   a. $6x + 8 = 7x + 13$

   b. $6x + 8 = 2(3x + 4)$

c. $6x + 8 = 6x + 13$

2.

a. $\frac{1}{4}(12 - 4x) = 3 - x$

b. $x - 3 = 3 - x$

c. $x - 3 = 3 + x$

3.

a. $-5x - 3x + 2 = -8x + 2$

b. $-5x - 3x - 4 = -8x + 2$

c. $-5x - 4x - 2 = -8x + 2$

4.

a. $4(2x - 2) + 2 = 4(x - 2)$

b. $4x + 2(2x - 3) = 8(x - 1)$

c. $4x + 2(2x - 3) = 4(2x - 2) + 2$

5.

a. $x - 3(2 - 3x) = 2(5x + 3)$

b. $x - 3(2 + 3x) = 2(5x - 3)$

c. $x - 3(2 - 3x) = 2(5x - 3)$

1. What do you notice about equations with one solution? How is this different from equations with no solutions and equations that are true for every x?

Student Response

1.

a. One solution. $x = -5$.

b. Infinitely many solutions, or true for every x.

c. No solution.

2.

a. Infinitely many solutions, or true for every x.

b. One solution. $x = 3$.

c. No solution.

3.

a. Infinitely many solutions, or true for every x.

b. No solutions.

c. One solution. $x = -4$.

4.

a. One solution. $x = -\frac{1}{2}$.

b. No solution.

c. Infinitely many solutions, or true for every x.

5.

a. No solution.

b. One solution. $x = 0$.

c. Infinitely many solutions, or true for every x.

1. Answers vary. Sample response: Equations with only one solution have a different amount of xs on each side, or the coefficients of x are not equal. Equations with no solution have the same coefficients of x but a different constant on each side, while equations with infinitely many solutions have equivalent expressions on each side of the equation.

Are You Ready for More?

Consecutive numbers follow one right after the other. An example of three consecutive numbers is 17, 18, and 19. Another example is -100, -99, -98.

1. Choose any set of three consecutive numbers. Find their mean. What do you notice?
2. Find the mean of another set of three consecutive numbers. What do you notice?
3. Explain why the thing you noticed must always work, or find a counterexample.

Student Response

Explanations vary. Sample response: Any three consecutive numbers can be represented as x, $x + 1$, and $x + 2$. Then the mean of the three numbers is $\frac{x+(x+1)+(x+2)}{3} = x + 1$.

Activity Synthesis

Review each of the equations and how many solutions it has. If students disagree, ask each to explain their thinking about the equation and work to reach agreement. Once students are satisfied with the solutions, display the following questions for all to see:

- "What do you notice about equations with one solution?"
- "What do you notice about equations with no solutions?"
- "What do you notice about equations with infinitely many solutions?"

Give students brief quiet think time and then ask them to share a response to at least one of the questions with a partner. After partners have shared, invite students to share something they noticed with the class. Record students' responses for all to see next to the relevant question.

Action and Expression: Internalise Executive Functions. Provide students with a graphic organiser to support their participation during the synthesis. Invite students to describe what to look for to determine whether an equation will have one solution, no solutions, or infinitely many solutions, and to include examples for each.
Supports accessibility for: Conceptual processing; Organisation Writing, Conversing: Stronger and Clearer Each Time. Use this routine for students to respond in writing to one of the three questions for whole-class discussion. Divide the class into thirds and assign each group of students one of the questions. Give students 3 minutes of quiet time to write a response. Invite students to meet with at least 2 other students to share and get feedback on their writing. Students should first meet with a partner that responded to the same question they did, before meeting with a student from a different group. Encourage listeners to ask clarifying questions such as, "Can you describe that using a different example?" or "What is another feature of that type of equation?" Invite the students to write a final draft based on their peer feedback. This will help students solidify their understanding of the number of solutions in a given equation by conversing with their partners.
Design Principle(s): Optimise output; Cultivate conversation

Lesson Synthesis

Instruct students to write three equations with a variable term and a constant term on each side of the equation. Their equations should be one with no solution, one with infinitely many solutions, and one with exactly one solution. When they think they have three equations that meet these requirements, tell students to trade with a partner, then identify which equation is each type. Give partners 2–3 minutes to check their solutions and discuss how they came up with their equations.

Ask students, "How did you know how to make each type of equation?" (I knew that the single-solution equation should have different coefficients for the variable terms, I knew that the many-solution equation should have equivalent expressions on each side, and I knew that the no-solution equation should only differ by a constant term on each side.)

If time allows, consider making a poster for permanent display that shows an equation with coefficient, variable, and constant terms emphasised in different colours.

8.4 How Does She Know?

Cool Down: 5 minutes

Student Task Statement

Elena began to solve this equation:

$$\frac{12x + 6(4x + 3)}{3} = 2(6x + 4) - 2$$
$$12x + 6(4x + 3) = 3(2(6x + 4) - 2)$$
$$12x + 6(4x + 3) = 6(6x + 4) - 6$$
$$12x + 24x + 18 = 36x + 24 - 6$$

When she got to the last line she stopped and said the equation is true for all values of x. How could Elena tell?

Student Response

Answers vary. Sample response: Elena could see that there are the same number of xs and the same constants on each side of the equation.

Student Lesson Summary

Sometimes it's possible to look at the structure of an equation and tell if it has infinitely many solutions or no solutions. For example, look at

$2(12x + 18) + 6 = 18x + 6(x + 7)$.

Using the distributive property on the left and right sides, we get

$24x + 36 + 6 = 18x + 6x + 42$.

From here, collecting like terms gives us

$24x + 42 = 24x + 42$.

Since the left and right sides of the equation are the same, we know that this equation is true for any value of x without doing any more moves!

Similarly, we can sometimes use structure to tell if an equation has no solutions. For example, look at

$6(6x + 5) = 12(3x + 2) + 12$.

If we think about each move as we go, we can stop when we realise there is no solution:

$\frac{1}{6} \times 6(6x+5)$	$= \frac{1}{6} \times (12(3x+2)+12)$	Multiply each side by $\frac{1}{6}$.
$6x+5$	$= 2(3x+2)+2$	Multiply through by $\frac{1}{6}$ on the right side.
$6x+5$	$= 6x+4+2$	Expand the bracket on the right side.

The last move makes it clear that the **constant terms** on each side, 5 and $4+2$, are not the same. Since adding 5 to an amount is always less than adding $4+2$ to that same amount, we know there are no solutions.

Doing moves to keep an equation balanced is a powerful part of solving equations, but thinking about what the structure of an equation tells us about the solutions is just as important.

Glossary

- coefficient
- constant term

Lesson 8 Practice Problems

1. **Problem 1 Statement**

   Lin was looking at the equation $2x - 32 + 4(3x - 2462) = 14x$. She said, "I can tell right away there are no solutions, because on the left side, you will have $2x + 12x$ and a bunch of constants, but you have just $14x$ on the right side." Do you agree with Lin? Explain your reasoning.

   Solution

   Lin is correct. Responses vary. Sample response: Ignoring everything but the terms with x on the left side, we have $2x$ and $4(3x)$. In total, this will give $14x$. All of the constant terms on the left side are negative, so they won't cancel to 0. Therefore, we have $14x + \text{non-zero stuff} = 4x$, which will have no solutions.

2. **Problem 2 Statement**

   Han was looking at the equation $6x - 4 + 2(5x + 2) = 16x$. He said, "I can tell right away there are no solutions, because on the left side, you will have $6x + 10x$ and a bunch of constants, but you have just $16x$ on the right side." Do you agree with Han? Explain your reasoning.

   Solution

   Han is incorrect. Responses vary. Sample response: Ignoring everything but the terms with x on the left side, we have $6x$ and $2(5x)$. In total, this will give $16x$. Collecting all the constant terms on the left side will give -4 + $2(2)$, which is 0. Therefore, we have $16x + 0 = 16x$, which is true for all values of x.

3. **Problem 3 Statement**

Decide whether each equation is true for all, one, or no values of x.

a. $6x - 4 = -4 + 6x$

b. $4x - 6 = 4x + 3$

c. $-2x + 4 = -3x + 4$

Solution

a. True for all values of x.

b. True for no values of x.

c. True for one value of x.

4. **Problem 4 Statement**

Solve each of these equations. Explain or show your reasoning.

a. $3(x - 5) = 6$

b. $2\left(x - \frac{2}{3}\right) = 0$

c. $4x - 5 = 2 - x$

Solution

a. $x = 7$. Explanations vary. Sample response: Multiply both sides by $\frac{1}{3}$, then add 5.

b. $x = \frac{2}{3}$. Explanations vary. Sample response: Multiply both sides by $\frac{1}{2}$, then add $\frac{2}{3}$.

c. $\frac{7}{5}$. Explanations vary. Sample response: Add x and 5 to both sides, then multiply by $\frac{1}{5}$.

5. **Problem 5 Statement**

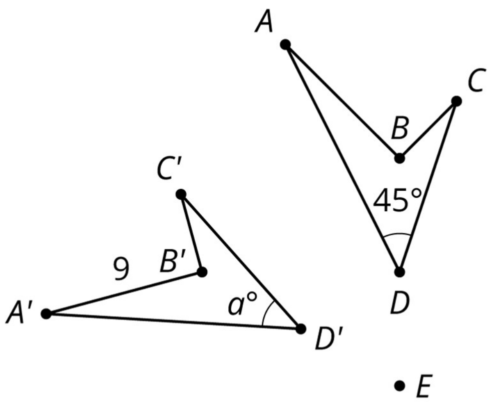
The points (-2,0) and (0,-6) are each on the graph of a linear equation. Is (2,6) also on the graph of this linear equation? Explain your reasoning.

Solution

No. Answers vary. Sample response: If the two points are graphed with the line that goes through both of them, the line does not pass through the first quadrant where (2,6) is plotted.

6. **Problem 6 Statement**

In the picture triangle $A'B'C'$ is an image of triangle ABC after a rotation. The centre of rotation is E.



a. What is the length of side AB? Explain how you know.

b. What is the size of angle D'? Explain how you know.

Solution

a. 9 units. Rotations preserve side lengths, and side $A'B'$ corresponds to side AB under this rotation.

b. 45 degrees. Rotations preserve angle size, and angles D and D' correspond to each other under this rotation.