MEI Mathematics Education Innovation

Lesson 4: Scaled relationships

Goals

- Explain (orally and in writing) that corresponding angles in a shape and its scaled copies have the same size.
- Identify (orally and in writing) corresponding distances or angles that can show that a shape is not a scaled copy of another.
- Recognise that corresponding distances in a shape and its scaled copy are related by the same scale factor as corresponding sides.

Learning Targets

- I can use corresponding distances and corresponding angles to tell whether one shape is a scaled copy of another.
- When I see a shape and its scaled copy, I can explain what is true about corresponding angles.
- When I see a shape and its scaled copy, I can explain what is true about corresponding distances.

Lesson Narrative

In previous lessons, students looked at the relationship between a shape and a scaled copy by finding the scale factor that relates the side lengths and by using tracing paper to compare the angles. This lesson takes both of these comparisons a step further.

- Students study *corresponding distances* between points that are not connected by line segments, in both scaled and unscaled copies. They notice that when a shape is a scaled copy of another, corresponding distances that are not connected by a line segment are also related by the same scale factor as corresponding sides.
- Students use protractors to test their observations about corresponding angles. They verify in several sets of examples that corresponding angles in a shape and its scaled copies are the same size.

Students use both insights—about angles and distances between points—to make a case for whether a shape is or is not a scaled copy of another. Practice with the use of protractors will help develop a sense for measurement accuracy, and how to draw conclusions from said measurements, when determining whether or not two angles are the same.

Addressing

- Solve problems involving scale drawings of geometric shapes, including calculating actual lengths and areas from a scale drawing and reproducing a scale drawing at a different scale.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- Stronger and Clearer Each Time
- Compare and Connect
- Discussion Supports
- Notice and Wonder

Required Materials

Geometry toolkits
tracing paper, graph paper, coloured pencils, scissors, and an index card to use as a straightedge or to mark right angles, plus a ruler and protractor. Clear protractors with no holes and with radial lines printed on them are recommended.

Required Preparation

Make sure students have access to their geometry toolkits, especially rulers and protractors.

Student Learning Goals

Let's find relationships between scaled copies.

4.1 Three Quadrilaterals (Part 1)

Warm Up: 5 minutes

This warm-up gives students a chance to practise identifying corresponding angles of scaled copies, measure angles using a protractor, and test their earlier conjecture that corresponding angles have the same size.

Instructional Routines

- Notice and Wonder

Launch

Have students look at the shapes in the activity, and ask "What do you notice? What do you wonder?" Call out in particular questions about the angles in the shapes (e.g., whether corresponding ones have the same size). Tell students that they will test their previous observation about the angles of scaled shapes, this time by using protractors instead of tracing paper.

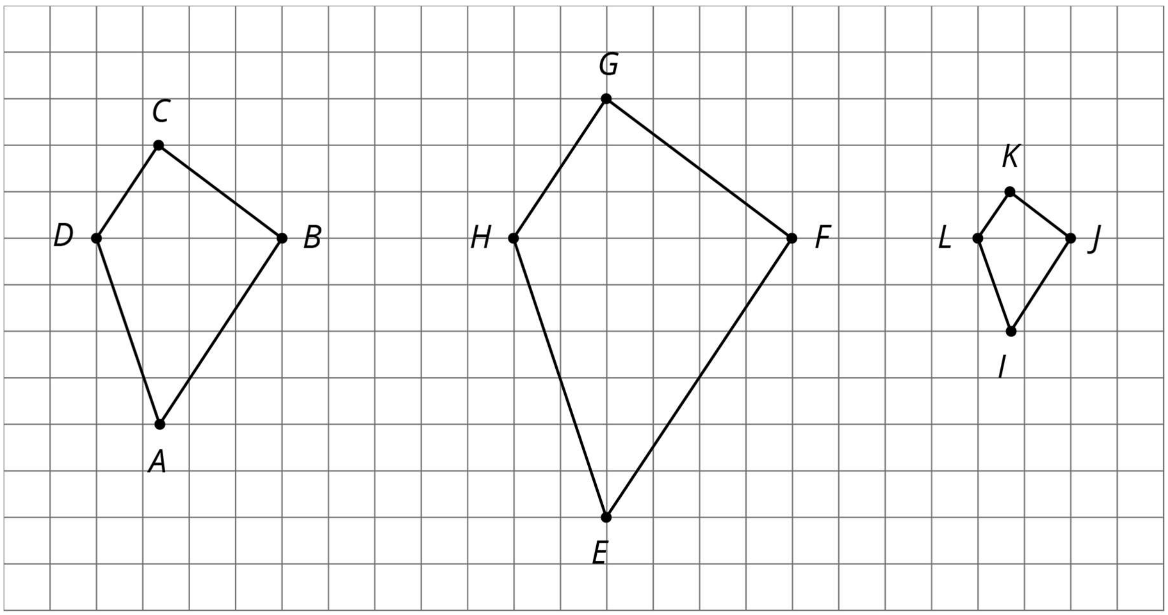
Provide access to protractors. Clear protractors with no holes and with radial lines printed on them are recommended here. Some angles may be challenging to measure because of the size of the polygons. If students find the sides of a polygon not long enough to accommodate angle measurements, suggest that they extend the lines, or demonstrate how to do so (especially if available protractors are opaque with holes in the middle).

Anticipated Misconceptions

Some students may read the wrong number on the protractor, moving down from the 180° mark instead of up from the 0° mark, or reading the measurement outside of one of the lines forming the angle instead of between the two lines. Clarify the angle being measured, how to line up the protractor, or how to read the markings correctly.

Student Task Statement

Each of these polygons is a scaled copy of the others.



1. Name two pairs of corresponding angles. What can you say about the sizes of these angles?
2. Check your prediction by measuring at least one pair of corresponding angles using a protractor. Record your measurements to the nearest 5°.

Student Response

1. Answers vary. Sample response:
   - Angles ABC and EFG (a.k.a. angles B and F)
   - Angles LIJ and HEF (a.k.a. angles I and E)

   The corresponding angles of the polygons will be the same size.

2. At least two angles from one of these lists:
   - Angles A, E, and I each measure about 50°.
   - Angles B, F, and J each measure about 95°.

- Angles C, G, and K each measure about 90°.
- Angles D, H, and L each measure about 125°.

Activity Synthesis

Select a few students to share their angle measurements and poll the class briefly for agreement and disagreement. Discuss major discrepancies, if any. Students should be able to confirm that all corresponding angles in the scaled polygons are equal.

If desired, ask students whether recording the angles to the nearest 1 degree would be appropriate: in general, the thickness of the line segments and the markings on the protractor limit accuracy, so reporting to the nearest 5 degrees is appropriate (as long as none of the angles are too close to halfway between two increments).

4.2 Three Quadrilaterals (Part 2)

10 minutes

Students have seen that the lengths of corresponding line segments in a shape and its scaled copy vary by the same scale factor. Here, they learn that in such a pair of shapes, *any* corresponding distances—not limited to lengths of sides or line segments—are related by the same scale factor. The side lengths of the polygons in this task cannot be easily determined, so students must look to other distances to compare.

Students must take care when they identify corresponding vertices and distances. As students work, urge them to attend to the order in which points or line segments are listed.

If students are not sure what to make out of the values in the table (for the second question), encourage them to consider the corresponding distances of two shapes at a time. For example, ask: What do you notice about the corresponding vertical distances in $IJKL$ and $EFGH$? What about the corresponding horizontal distances in those two shapes?

Instructional Routines

- Compare and Connect

Launch

Arrange students in groups of 2. Ask if they can tell the lengths of line segments GF or DC from the grid (without using rulers). Explain that they will explore another way to compare length measurements in scaled copies.

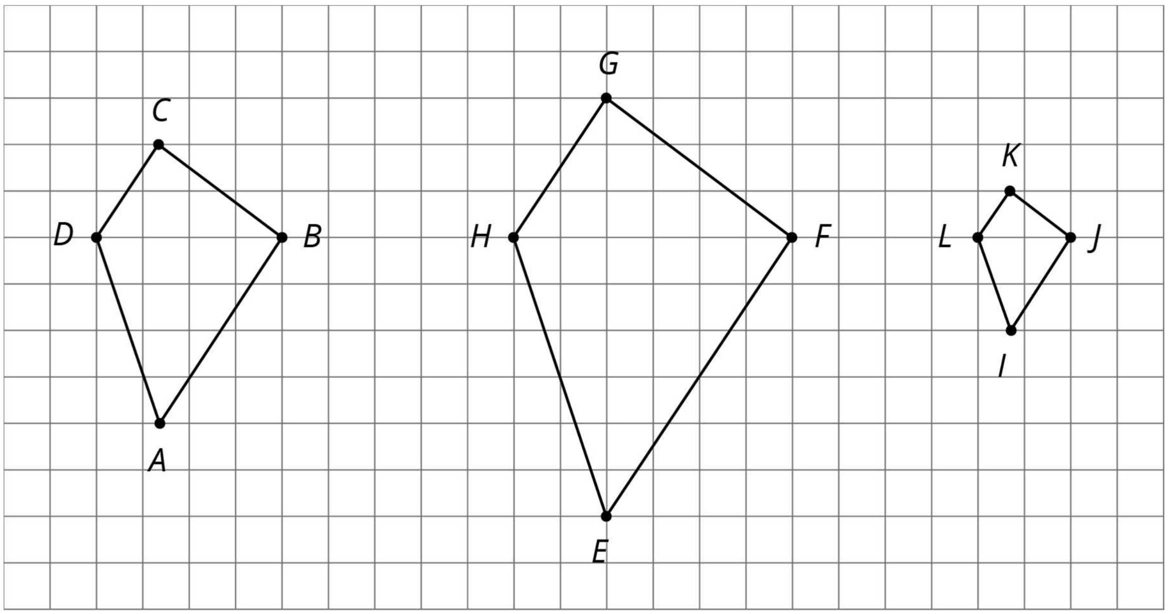
Give students 2–3 minutes of quiet work time for the first two questions, and 1 minute to discuss their responses with a partner before continuing on to the last question.

Anticipated Misconceptions

Students may list the corresponding vertices for distances in the wrong order. For example, instead of writing LJ as the distance corresponding to DB, they may write JL. Remind students of the corresponding points by asking, “Which vertex in $IJKL$ corresponds to D? Which corresponds to B?” and have them match the order of the vertices accordingly.

Student Task Statement

Each of these polygons is a scaled copy of the others. You already checked their corresponding angles.



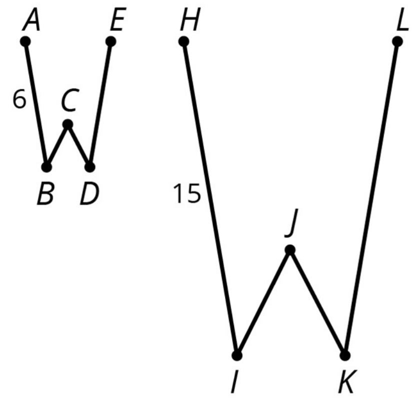
1. The side lengths of the polygons are hard to tell from the grid, but there are other *corresponding distances* that are easier to compare. Identify the distances in the other two polygons that correspond to DB and AC, and record them in the table.

quadrilateral	distance that corresponds to DB	distance that corresponds to AC
$ABCD$	$DB = 4$	$AC = 6$
$EFGH$		
$IJKL$		

2. Look at the values in the table. What do you notice?

   Pause here so your teacher can review your work.

3. The larger shape is a scaled copy of the smaller shape.



a. If $AE = 4$, how long is the corresponding distance in the second shape? Explain or show your reasoning.

b. If $IK = 5$, how long is the corresponding distance in the first shape? Explain or show your reasoning.

Student Response

1.

quadrilateral	distance that corresponds to DB	distance that corresponds to AC
$ABCD$	$DB = 4$	$AC = 6$
$EFGH$	$HF = 6$	$EG = 9$
$IJKL$	$LJ = 2$	$IK = 3$

2. These corresponding distances are related by the same scale factor even though they are not side lengths.

3.

a. $HL = 10$. Sample explanation: $15 \div 6 = 2.5$, so the second shape is related to the first shape by a scale factor of 2.5. HL is the corresponding distance to AE and is also related by a factor of 2.5. $(2.5) \times 4 = 10$.

b. $BD = 2$. Sample explanation: The scale factor from the small shape to the larger copy is 2.5, so dividing KI by 2.5 gives the corresponding distance in the original shape. $5 \div 2.5 = 2$.

Activity Synthesis

Display the completed tables for all to see. To highlight how all distances in a scaled copy (not just the side lengths of the shape) are related by the same scale factor, discuss:

- How does the vertical distance in $ABCD$ compare to that in $EFGH$? How do the horizontal distances in the two polygons compare? Do the pairs of vertical and horizontal distances share the same scale factor?

- How do the vertical distances in $EFGH$ and $IJKL$ compare? What about the horizontal distances? Is there a common scale factor? What is that scale factor?
- What scale factor relates the corresponding lengths and distances in the two drawings of the letter W?

Representation: Internalise Comprehension. Demonstrate and encourage students to use colour coding and annotations to highlight connections between shapes. For example, mark corresponding line segments using the same colour.
Supports accessibility for: Visual-spatial processing Speaking: Compare and Connect. Use this routine to call attention to the different ways students may identify scale factors. Display the following statements: "The scale factor from $EFGH$ to $IJKL$ is 3," and "The scale factor from $EFGH$ to $IJKL$ is $\frac{1}{3}$." Give students 2 minutes of quiet think time to read and consider whether either or both of the statements are correct. Invite students to share their initial thinking with a partner before selecting 2–3 students to share with the class. In this discussion, listen for and amplify any comments that refer to the order of the original shape and its scaled copy, as well as those who identify corresponding vertices and distances. Draw students' attention to the different ways to describe the relationships between scaled copies and the original shape.
Design Principle(s): Maximise meta-awareness

4.3 Scaled or Not Scaled?

10 minutes

The purpose of this activity is for students to determine that shapes are not scaled copies, even though they have either corresponding angles with equal sizes or corresponding distances multiplied by the same scale factor. This shows that to determine one shape is a scaled copy of another, we have to check *both* the corresponding angles and the corresponding distances.

As students work, monitor for convincing arguments about why one polygon is or is not a scaled copy of the other. Ask them to present their cases during the whole-class discussion.

Instructional Routines

- Stronger and Clearer Each Time

Launch

Keep students in the same groups. Provide access to geometry toolkits. Give students 6–7 minutes of quiet work time.

Action and Expression: Internalise Executive Functions. Chunk this task into more manageable parts to support students who benefit from support with organisational skills in problem solving. For example, present one question at a time and monitor students to ensure they are making progress throughout the activity.
Supports accessibility for: Organisation; Attention

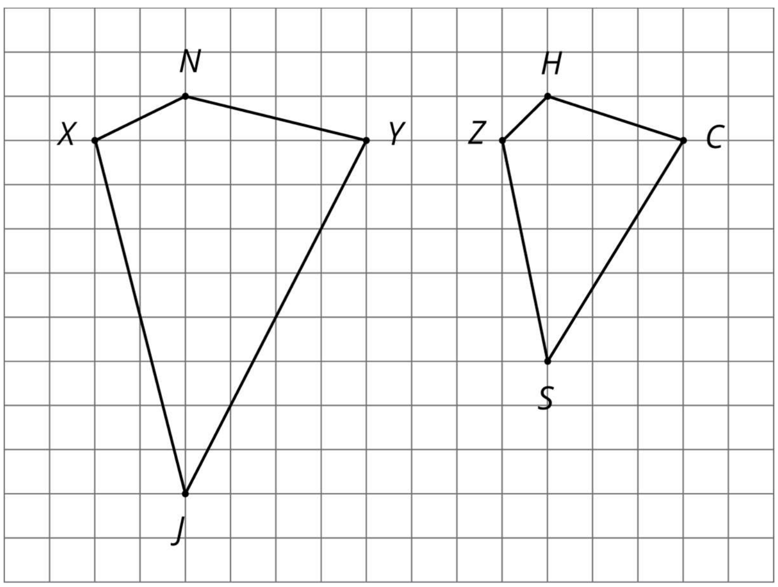
Anticipated Misconceptions

Students may rely on the appearance of the shapes rather than analyse given information to draw conclusions about scaling. Urge them to look for information about distances and angles (and to think about which tools could help them find such information) to support their argument.

Some students may struggle with comparing the corresponding angles in the first pair of shapes. Remind students of the tools that are at their disposal, and that they could extend the sides of the polygons, if needed, to make it easier to measure the angles.

Student Task Statement

Here are two quadrilaterals.



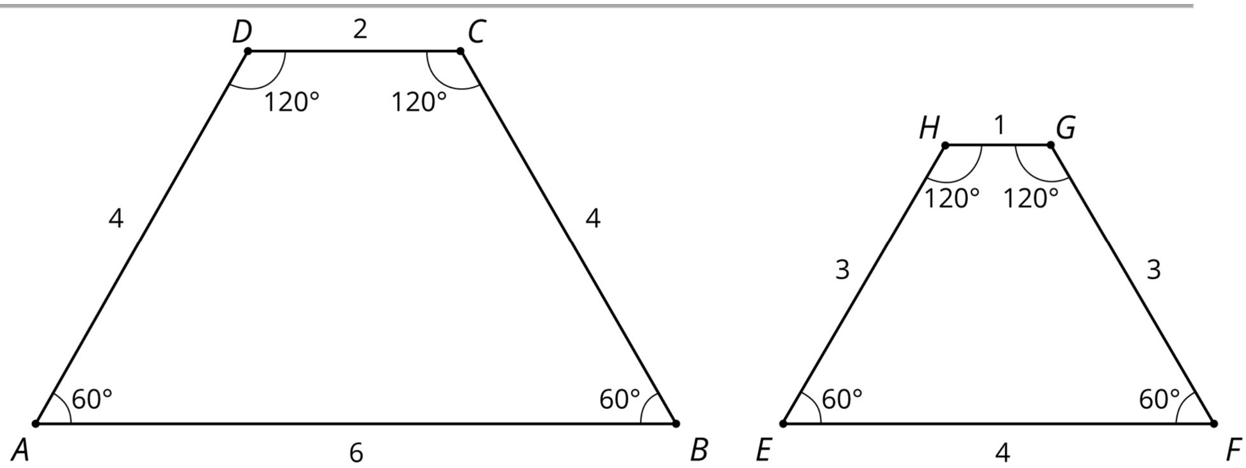
1. Mai says that polygon $ZSCH$ is a scaled copy of polygon $XJYN$, but Noah disagrees. Do you agree with either of them? Explain or show your reasoning.

2. Record the corresponding distances in the table. What do you notice?

quadrilateral	horizontal distance	vertical distance
$XJYN$	$XY =$	$JN =$
$ZSCH$	$ZC =$	$SH =$

3. Measure at least three pairs of corresponding angles in $XJYN$ and $ZSCH$ using a protractor. Record your measurements to the nearest 5°. What do you notice?

4. Here are two more quadrilaterals.



Kiran says that polygon $EFGH$ is a scaled copy of $ABCD$, but Lin disagrees. Do you agree with either of them? Explain or show your reasoning.

Student Response

1. Answers vary. Sample response: Noah is correct, because the corresponding angles are not equal. Mai may have noticed that the corresponding distances are multiplied by $\frac{3}{2}$ and thought this meant the polygons are similar.

2.

quadrilateral	horizontal distance	vertical distance
$XJYN$	$XY = 6$	$JN = 9$
$ZSCH$	$ZC = 4$	$SH = 6$

3. The corresponding angles are not all the same size. Rounded to the nearest 5°, the sizes are:

$XJYN$	$ZSCH$
angle X measures 100°	angle Z measures 125°
angle J measures 40°	angle S measures 40°
angle Y measures 75°	angle C measures 75°
angle N measures 140°	angle H measures 115°

Since the corresponding angles are not equal, the polygons are definitely not scaled copies of one another.

4. Answers vary. Sample response: Lin is correct, because the corresponding distances are not multiplied by the same number (compared to $ABCD$, the top side in $EFGH$ is half as long, while the bottom side is two-thirds as long). Kiran may have noticed that the corresponding angles are equal and thought this meant the polygons are similar. I noticed that the scale factors for the corresponding sides are not the same. AB and EF are related by a scale factor of $\frac{2}{3}$, but DC and HG are related by a scale factor of $\frac{1}{2}$.

Are You Ready for More?

All side lengths of quadrilateral $MNOP$ are 2, and all side lengths of quadrilateral $QRST$ are 3. Does $MNOP$ have to be a scaled copy of $QRST$? Explain your reasoning.

Student Response

No. $MNOP$ could be a square and $QRST$ could be a rhombus that is not a square. Since the angles are different, $MNOP$ is not a scaled copy of $QRST$.

Activity Synthesis

The goal of this discussion is to make clear that angle measurements and distances are both important when deciding whether two polygons are scaled copies. To highlight the different arguments about whether one polygon is a scaled copy of another, consider debriefing with a role play. Ask four students to take on the roles of the four characters—Mai and Noah in the first question, and Kiran and Lin in the second—and make a brief argument about why they believe one shape is a scaled copy of the other in each case. Poll the class after each pair of cases are presented and find out with whom students agree.

Writing, speaking, and representing: Stronger and Clearer Each Time. Use this routine to help students improve their writing, by providing them with multiple opportunities to clarify their explanations through conversation. At the appropriate time, give students time to meet with 2–3 partners to share and get feedback on their written justification for whether they agree with Mai or Noah. Display prompts for feedback that students can use to help their partner strengthen and clarify their ideas. For example, "Your explanation tells me . . .", "Can you say more about why you . . . ?", and "A detail (or word) you could add is _____, because . . . ." Give students with 3–4 minutes to revise their initial draft based on feedback from their peers.
Design Principle(s): Optimise output (for explanation)

4.4 Comparing Pictures of Birds

Optional: 10 minutes (there is a digital version of this activity)

In this activity, students use what they know about corresponding lengths and angles to show that one picture of a bird is *not* a scaled copy of the other. Unlike in previous tasks, minimal scaffolding is given here, so students need to decide what evidence is necessary to explain or show an absence of scaling. As students work, notice the different ways students use corresponding lengths and angles to think about scaling.

Identifying a specific measurement in one image and the corresponding measurements in the other (which show that one is not a scaled copy of the other) requires thinking about the structure of the pictures.

Monitor students for different approaches to show that one image is not a scaled copy of the other:

- Identifying corresponding angles in the two images which have different sizes

- Identifying two pair of corresponding line segments in the images with different scale factors

Select students who use these methods and ask them to present their approaches, in this sequence, during the discussion.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- Discussion Supports

Launch

Arrange students in groups of 3–4. Provide access to geometry toolkits.

Give students 2 minutes of quiet work time followed by 1–2 minutes to discuss in groups how to find evidence that one picture is not a scaled copy of the other. Then, briefly discuss students' ideas as a class. Consider listing the ideas for all to see and discussing which ones are likely to be most effective.

If students' ideas deviate from drawing corresponding points and line segments and comparing distances and angles, guide them with some prompts. For example:

- Pick a point that can be easily referenced (e.g., the tip of one wing) on one picture. Ask for the corresponding point on the other.
- Ask if that pair of corresponding points could help them determine if one picture is scaled from the other. If not, ask what else might be needed.
- Add another point and a line segment connecting the two points. Ask if or how the line segment could help, and so on.

Anticipated Misconceptions

Students may draw two line segments that do not share a point, or choose non-corresponding points and line segments on the two shapes. Refer students to earlier work involving polygons and point out pairs of distances that could be used for comparison and those that could not be. Remind them that we can only compare the corresponding parts, not just any two parts.

Student Task Statement

Here are two pictures of a bird. Find evidence that one picture is not a scaled copy of the other. Be prepared to explain your reasoning.

Student Response

Answers vary. Sample reasoning:

- Corresponding angles do not match, so one picture cannot be a scaled copy of the other.
- The lengths of corresponding line segments are not related by the same scale factor, so one picture cannot be a scaled copy of the other.

Activity Synthesis

Invite select students to share their reasoning. Ask how they knew that certain pairs of points in the images were corresponding points. Students could identify a unique feature such as the eyes or the tip of the wing. Make sure students understand that:

- Corresponding angles on an image and a scaled copy have the same size
- Corresponding line segments on an image and a scaled copy are related by the same scale factor

To conclude that an image is *not* a scaled copy of another, it is sufficient to find one pair of corresponding angle sizes that are different or one pair of corresponding line segments with different lengths.

Speaking: Discussion Supports. Give students additional time to make sure that everyone in their group can explain or justify why the copy of the bird is not a scaled copy. Then, vary who is called on to represent the ideas of each group. This routine will prepare students for the role of group representative and to support each other to take on that role.
Design Principle(s): Optimise output (for explanation)

Lesson Synthesis

- Does a scale factor affect any other measurements other than line segment lengths?
- How can we be sure that a shape is a scaled copy? What features do we check?

When a scaled copy is created from a shape, we know that:

- The distances between any two points in the original shape, even those not connected by line segments, are scaled by the same scale factor.
- The corresponding angles in the original shape and scaled copies are congruent.

Polygons are a perfect context in which to apply these two ideas, being made up of line segments meeting at angles. So we can use these observations to check whether a polygon is actually a scaled copy of another. If all the corresponding angles are the same size and all corresponding distances are all scaled by the same factor, then we can conclude that it is a scaled copy of the other.

4.5 Corresponding Polygons

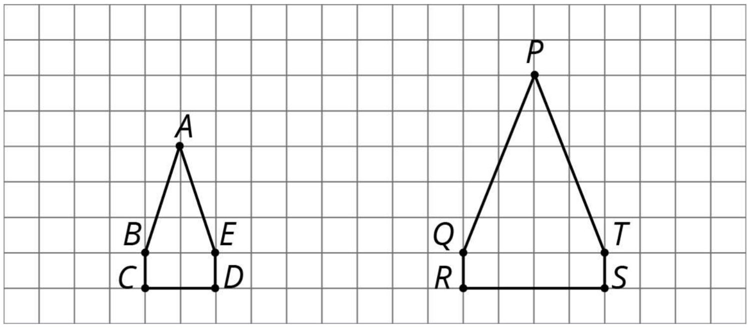
Cool Down: 5 minutes

Launch

Provide access to geometry toolkits.

Student Task Statement

Here are two polygons on a grid.



Is $PQRST$ a scaled copy of $ABCDE$? Explain your reasoning.

Student Response

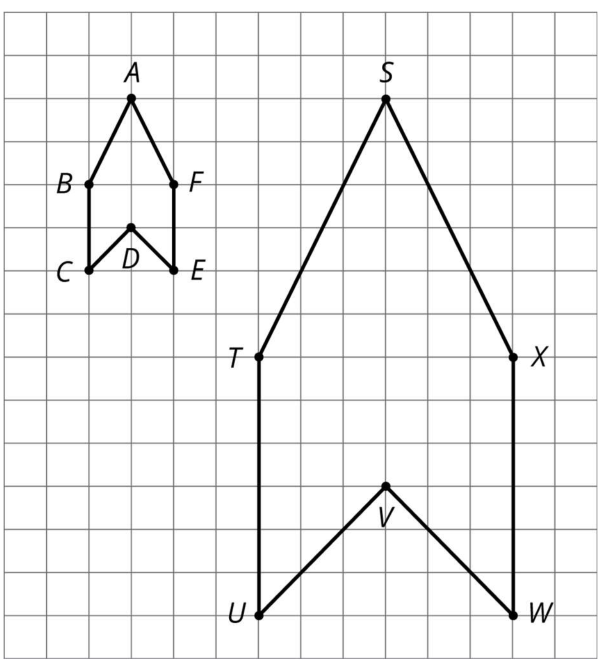
No. Sample explanation: $PQRST$ is not a scaled copy of $ABCDE$ because we need to use different scale factors when comparing corresponding lengths (1 for corresponding line segments BC and QR and 2 for corresponding line segments CD and RS). Also, not all of their corresponding angles are the same size. Angle A and angle P are not the same size.

Student Lesson Summary

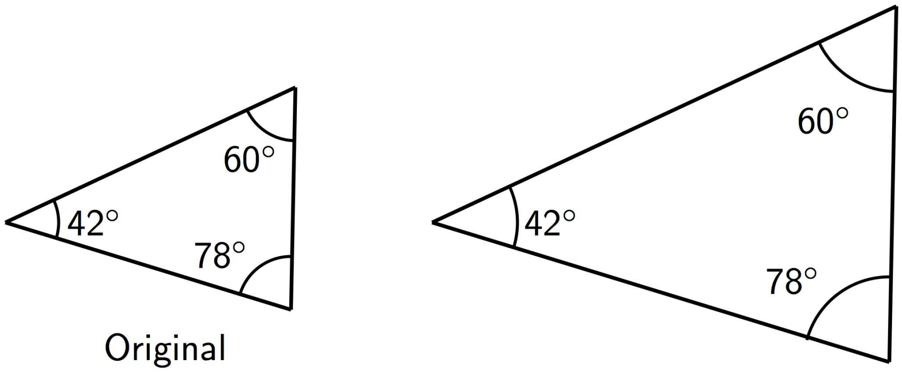
When a shape is a scaled copy of another shape, we know that:

- All distances in the copy can be found by multiplying the *corresponding distances* in the original shape by the same scale factor, whether or not the endpoints are connected by a line segment.

  For example, polygon $STUVWX$ is a scaled copy of polygon $ABCDEF$. The scale factor is 3. The distance from T to X is 6, which is three times the distance from B to F.

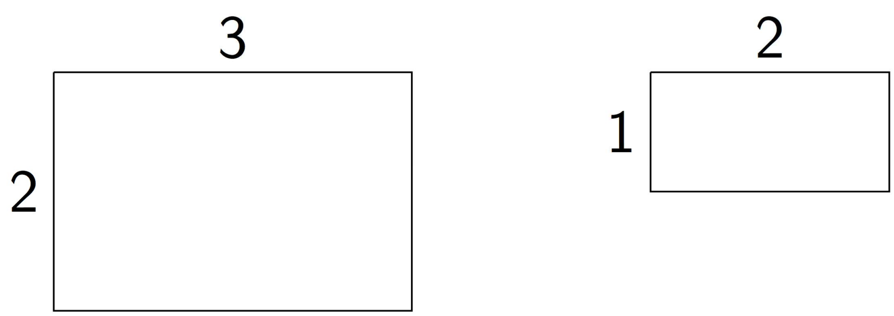


- All angles in the copy have the same size as the corresponding angles in the original shape, as in these triangles.



These observations can help explain why one shape is *not* a scaled copy of another.

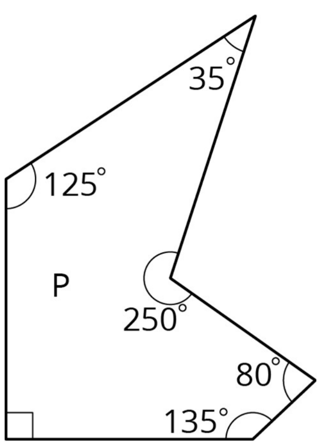
For example, even though their corresponding angles have the same size, the second rectangle is not a scaled copy of the first rectangle, because different pairs of corresponding lengths have different scale factors, $2 \times \frac{1}{2} = 1$ but $3 \times \frac{2}{3} = 2$.



Lesson 4 Practice Problems

Problem 1 Statement

Select **all** the statements that must be true for *any* scaled copy, Q, of polygon P.



a. The side lengths are all whole numbers.

b. The angle sizes are all whole numbers.

c. Q has exactly 1 right angle.

d. If the scale factor between P and Q is $\frac{1}{5}$, then each side length of P is multiplied by $\frac{1}{5}$ to get the corresponding side length of Q.

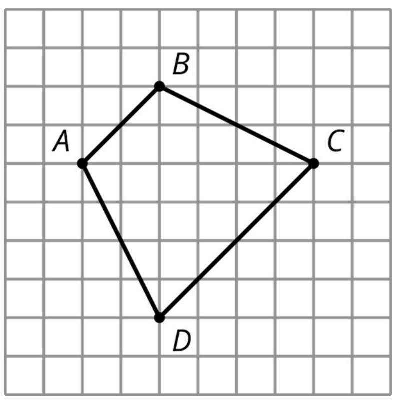
e. If the scale factor is 2, each angle in P is multiplied by 2 to get the corresponding angle in Q.

f. Q has 2 acute angles and 3 obtuse angles.

Solution ["B", "C", "D", "F"]

Problem 2 Statement

Here is quadrilateral $ABCD$.



Quadrilateral $PQRS$ is a scaled copy of quadrilateral $ABCD$. Point P corresponds to A, Q to B, R to C, and S to D.

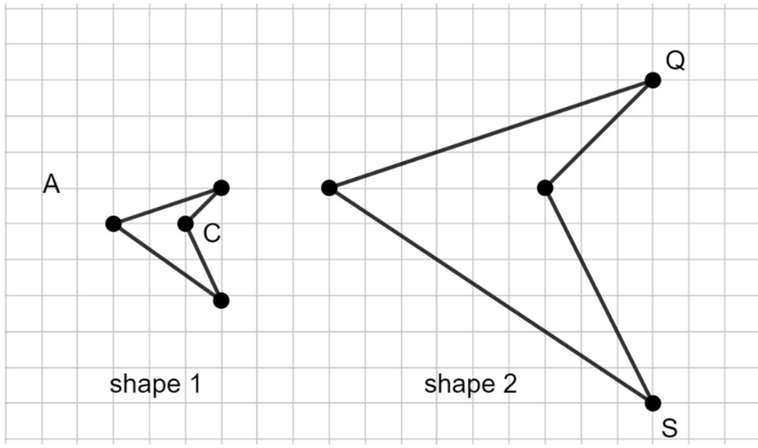
If the distance from P to R is 3 units, what is the distance from Q to S? Explain your reasoning.

Solution

Since the lengths of AC and BD are 6, and AC corresponds to PR, the scale factor must be $\frac{1}{2}$. Since QS corresponds to BD, QS must also be 3 units long.

Problem 3 Statement

Shape 2 is a scaled copy of shape 1.



a. Identify the points in shape 2 that correspond to the points A and C in shape 1. Label them P and R. What is the distance between P and R?

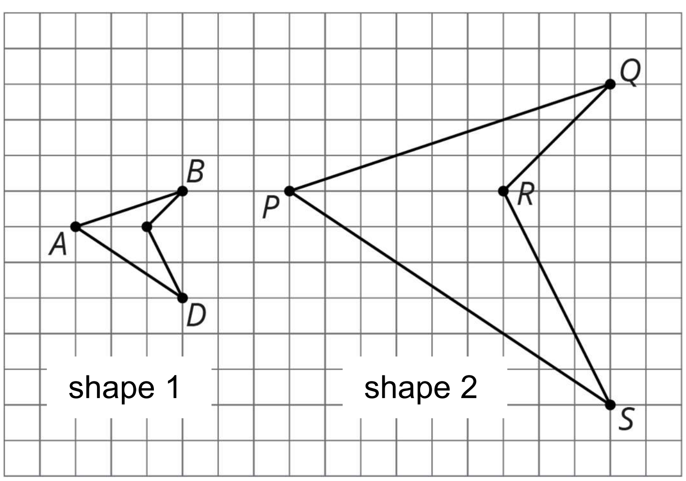
b. Identify the points in shape 1 that correspond to the points Q and S in shape 2. Label them B and D. What is the distance between B and D?

c. What is the scale factor that takes shape 1 to shape 2?

d. G and H are two points on shape 1, but they are not shown. The distance between G and H is 1. What is the distance between the corresponding points on shape 2?

Solution

a.



6 units

b. 3 units

c. 3 because distances between points in shape 2 are three times the corresponding distances in shape 1

d. 3 units because the scale factor is 3

Problem 4 Statement

To make 1 batch of lavender paint, the ratio of cups of pink paint to cups of blue paint is 6 to 5. Find two more ratios of cups of pink paint to cups of blue paint that are equivalent to this ratio.

Solution

Answers vary. Sample response: 12 cups of pink paint to 10 cups of blue paint and 18 cups of pink paint to 15 cups of blue paint. This is 2 batches and 3 batches, respectively, of this shade of lavender paint.