Approximating the Circumference and Area of a Circle

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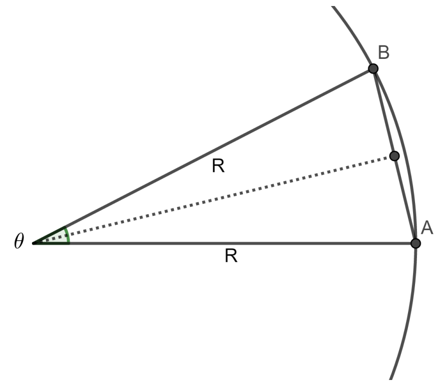


Consider a series of isosceles triangles of equal size placed inside a circle, with their vertices all at the circle center. To find the area of one of these triangles we will need the length of the line segment (a chord) that connects the two radial-line intersection points (points A and B in the figure above). Using the law of cosines, the length l_n of the n-th chord will be

$$l_n^2 = R^2 + R^2 - 2R^2 \cos(\theta_n)$$

from which

$$l_n = R\sqrt{2\left(1 - \cos(\theta_n)\right)}$$

where the angle θ_n is just $2\pi / n$, and n is the number of triangles placed inside the circle ($n \geq 3$). If we accumulate these (equal) lengths around the circle, especially as n increases, we will have an estimate of the **circumference of the circle**:

$$C \approx n\ R\sqrt{2\left(1 - \cos\left(\frac{2\pi}{n}\right)\right)} \tag{1}$$

Further, since we know the circumference of a circle is $2\,\pi\,R$, from Eq(1) it is to be expected that

$$\lim_{n\to\infty}\left\{\ n\sqrt{2\left(1 - \cos\left(\frac{2\pi}{n}\right)\right)}\ \right\} = 2\,\pi$$

and it can be shown, with some effort involving L'Hospital's Rule, that this is the case. However, without resorting to calculus we can simply plot this function (see below) and observe that it does in fact approach 2π as n increases.

Next, using one of the two right triangles formed inside the original isosceles triangle when the vertex angle θ_n is bisected (dotted line, above), we see that the area of the n-th whole original triangle will be twice that of the right triangle, so that

$$A_n = 2\left(\frac{1}{2}b_n\,h_n\right) \Rightarrow 2\left\{\frac{1}{2}\left(\frac{l_n}{2}R\cos\left(\frac{\theta_n}{2}\right)\right)\right\} = \frac{l_n}{2}R\cos\left(\frac{\theta_n}{2}\right)$$

and then using the result for l_n we have

$$A_n = \frac{R}{2}\sqrt{2\left(1-\cos\left(\theta_n\right)\right)}\left\{R\cos\left(\frac{\theta_n}{2}\right)\right\} = \frac{R^2}{2}\sqrt{2\left(1-\cos\left(\theta_n\right)\right)}\sqrt{\frac{1+\cos\left(\theta_n\right)}{2}}$$

using a trig identity. From this

$$A_n = \frac{R^2}{2}\sqrt{1-\cos^2\left(\theta_n\right)} = \frac{R^2}{2}\sin\left(\frac{2\pi}{n}\right)$$

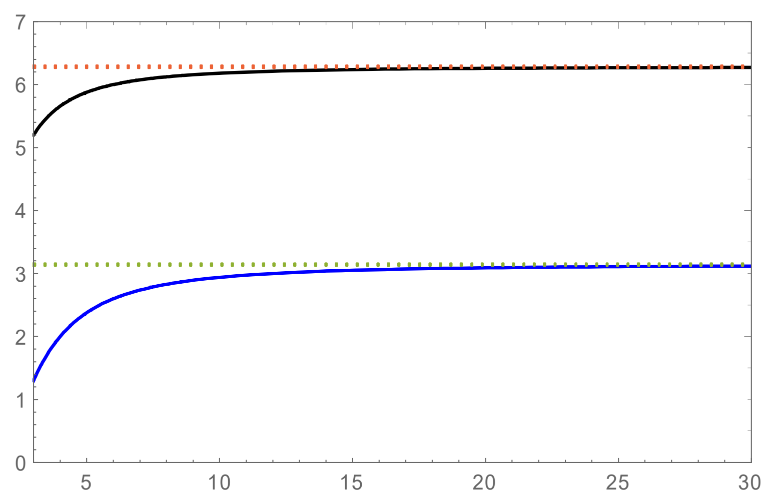
Then an approximation for the **area of the circle** is the sum of the n equal isosceles triangle areas, so that

$$\boxed{A_{circle} \approx n\,A_n = \frac{n}{2}\sin\left(\frac{2\pi}{n}\right)R^2} \qquad (2)$$

Since we know the formula for the area of the circle, as with the circumference Eq(1) we might expect that

$$\lim_{n\to\infty}\left\{\frac{n}{2}\sin\left(\frac{2\pi}{n}\right)\right\} = \pi$$

and this can be shown to be true. Even without a formal limit evaluation, we can see that, as n increases and the argument of the sine becomes small, the sine of that small angle approximately equals the angle (in radians). This immediately yields π as the expected value for this quantity, for "large" values of n. Further, the plot below shows Eq(1) (red) and Eq(2) (blue) as n increases; they are seen to approach the respective limits indicated above.



If we inscribe regular polygons in a circle, those polygons can be viewed as a set of adjacent interior isosceles triangles, one for each side of the polygon, with their vertices in common at the circle center. If we have from some other source the area of a polygon (Geogebra provides this), then an approximation for π can be found by dividing that polygon area by R^2, since Eq(2) is that polygon area, and dividing by the squared radius leaves the quantity that approaches π as n increases. This approximation will improve with increasing n, as the polygon side lengths decrease, and they get closer and closer to the circle boundary.

The process given by Eq(2) could be viewed as a sort of "Riemann sum" for finding the area of a circle, where the differential elements are triangles rather than rectangles.