Lesson 1: The areas of squares and their side lengths

Goals

- Calculate the area of a tilted square on a grid by using decomposition, and explain (orally) the solution method.
- Estimate the side length of a square by comparing it to squares with known areas, and explain (orally) the reasoning.

Learning Targets

- I can find the area of a tilted square on a grid by using methods like "decompose and rearrange" and "surround and subtract."
- I can find the area of a triangle.

Lesson Narrative

Students know from work in previous years how to find the area of a square given the side length. In this lesson, we lay the groundwork for thinking in the other direction: if we know the area of the square, what is the side length? Before students define this relationship formally in the next lesson, they estimate side lengths of squares with known areas using tools such as rulers and tracing paper. They also review key strategies for finding area that they encountered in earlier years that they will use to understand and explain informal proofs of Pythagoras' theorem.

In the warm-up, students compare the areas of figures that can easily be determined by either composing and counting square units or decomposing the figures into simple, familiar shapes. In the next activity, students find areas of "tilted" squares by enclosing them in larger squares whose areas can be determined and then subtracting the areas of the extra triangles. The next activity reinforces the relationship between the areas of squares and their side lengths, setting the stage for the definition of a square root in the next lesson.

Building On

- Write and evaluate numerical expressions involving whole-number exponents.
- Find the area of right triangles, other triangles, special quadrilaterals, and polygons by composing into rectangles or decomposing into triangles and other shapes; apply these techniques in the context of solving real-world and mathematical problems.

Addressing

- Use rational approximations of irrational numbers to compare the size of irrational numbers, locate them approximately on a number line diagram, and estimate the value of expressions (e.g., π^2). For example, by truncating the decimal expansion of $\sqrt{2}$, show that $\sqrt{2}$ is between 1 and 2, then between 1.4 and 1.5, and explain how to continue on to get better approximations.

Building Towards

- Use square root and cube root symbols to represent solutions to equations of the form $x^2 = p$ and $x^3 = p$, where p is a positive rational number. Evaluate square roots of small perfect squares and cube roots of small perfect cubes. Know that $\sqrt{2}$ is irrational.
- Understand and apply Pythagoras' theorem.
- Explain a proof of Pythagoras' theorem and its converse.
- Use rational approximations of irrational numbers to compare the size of irrational numbers, locate them approximately on a number line diagram, and estimate the value of expressions (e.g., π^2). For example, by truncating the decimal expansion of $\sqrt{2}$, show that $\sqrt{2}$ is between 1 and 2, then between 1.4 and 1.5, and explain how to continue on to get better approximations.

Instructional Routines

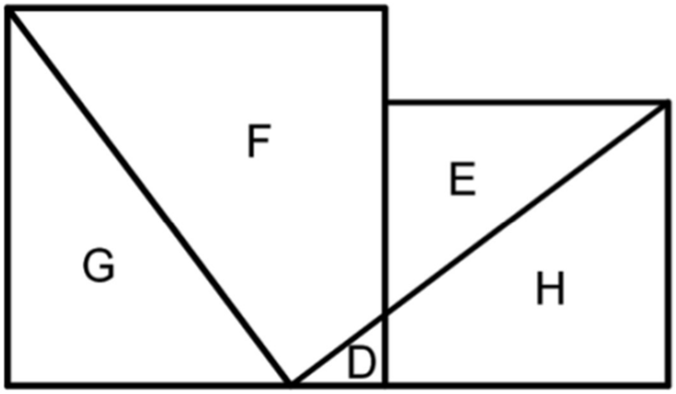
- Anticipate, Monitor, Select, Sequence, Connect
- Co-Craft Questions
- Compare and Connect
- Discussion Supports
- Think Pair Share

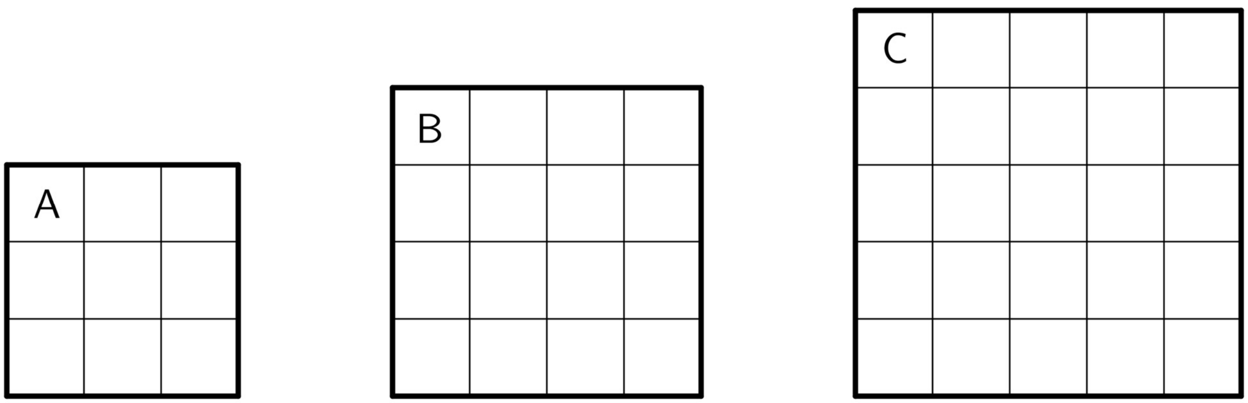
Required Materials

Geometry toolkits

tracing paper, graph paper, coloured pencils, scissors, and an index card to use as a straightedge or to mark right angles, a ruler and protractor. Clear protractors with no holes and with radial lines printed on them are recommended.

Pre-printed slips, cut from copies of the blackline master





Required Preparation

1 copy of the Making Squares blackline master, pre-cut, for every 2 students. These pieces are used again in the lesson "*A proof of Pythagoras' theorem*" and should be saved.

Student Learning Goals

Let's investigate the squares and their side lengths.

1.1 Two Regions

Warm Up: 5 minutes

The purpose of this warm-up is for students to review how to find the area of a region on a grid by decomposing and rearranging pieces. In the following activities, students will use these techniques to check area estimates made when approximating the value of the side length of a square used to calculate the area by squaring. They will also use these techniques to understand and explain a proof of Pythagoras' theorem.

As students work, identify students who used different strategies for finding area, including putting pieces together to make whole units in different ways and adding up the whole grid then subtracting the white space to indirectly figure out the area of the shaded region.

Launch

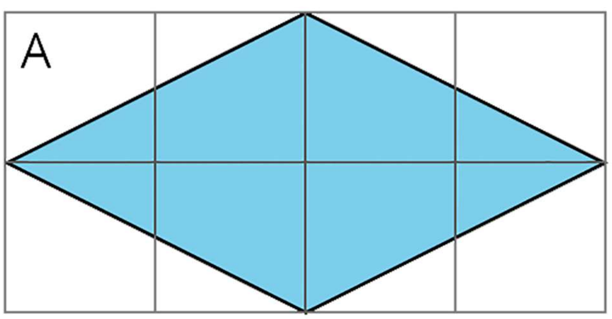
Tell students they are comparing the area of regions on two grids with the same unit size. Poll students on which region they predict is the larger area. Give students 2 minutes of quiet work time followed by a whole-class discussion.

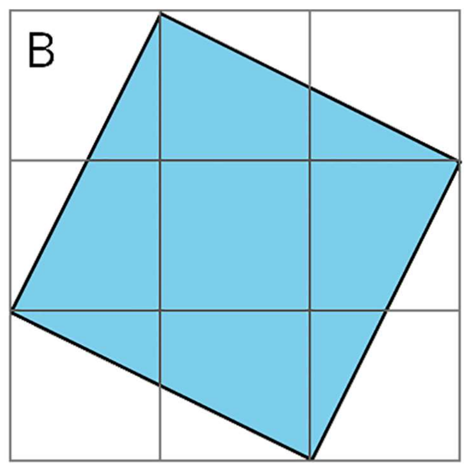
Anticipated Misconceptions

Students may think that the side length of either quadrilateral is 2 units because the sides are partitioned into two sub-pieces that look about 1 unit long. If so, point out that they do not lay along grid lines, they can use a compass or tracing paper to compare the side lengths to 2 units on the grid.

Student Task Statement

Which shaded region is larger? Explain your reasoning.





Student Response

Figure B has the larger shaded region because Figure A is 4 square units and Figure B is 5 square units.

Activity Synthesis

Poll students on which shaded region is the larger area. Select previously identified students to share their strategies for finding area. Record and display their responses for all to see.

If time allows, ask students to estimate the side lengths of each shaded region to prepare them for the work they will do in the next activity.

1.2 Decomposing to Find Area

15 minutes

The purpose of this task is for students to find the areas of squares whose side lengths are not easy to determine by inspection. The squares are represented in an increasingly abstract way: the first square shows all of the square units explicitly, making it easy to see that putting two "leftover" triangles together will make a rectangle with the same dimensions, or even piecing individual square units together that can be counted. The next only shows the grid units along the edges, providing a bridge to the third, which only provides labels for side lengths. The numbers are chosen to encourage the decomposing, rearranging, and subtracting strategy for finding the area, although decomposing and using the formulas for the area of a square and a triangle is also perfectly appropriate.

This sequence of tasks also provides practice for students in using the area and finding strategies they will need to understand and explain a proof of Pythagoras' theorem in a later lesson.

Identify students who find the area of the extra area in the triangles by:

- composing two triangles to make a rectangle.
- using the formula for the area of a triangle and multiplying the result by four.

Be sure these strategies are made explicit for the first square to help students who are only able to count unit squares make a successful transition to the third square. While decomposing the square and rearranging the pieces is a correct strategy, students need the subtraction strategy for more abstract work later, so it is better not to highlight the decomposing strategy that does not generalise as well.

Instructional Routines

- Compare and Connect

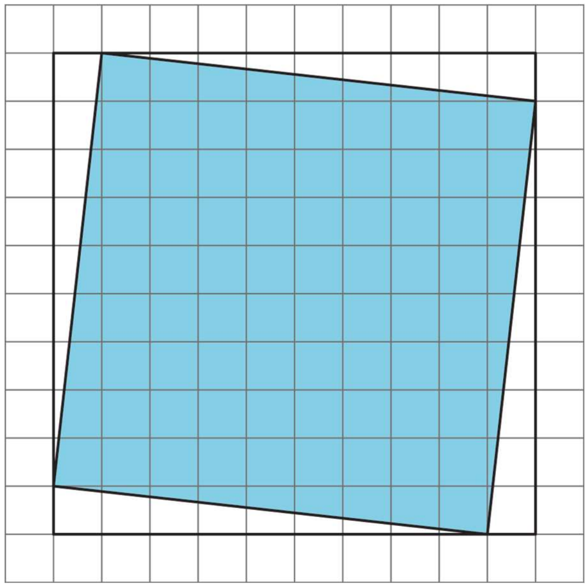
Launch

Give 2–3 minutes quiet work time and then ask students to pause and select 2–3 previously identified students to share how they found the area of the first square. Work time to finish the remaining questions.

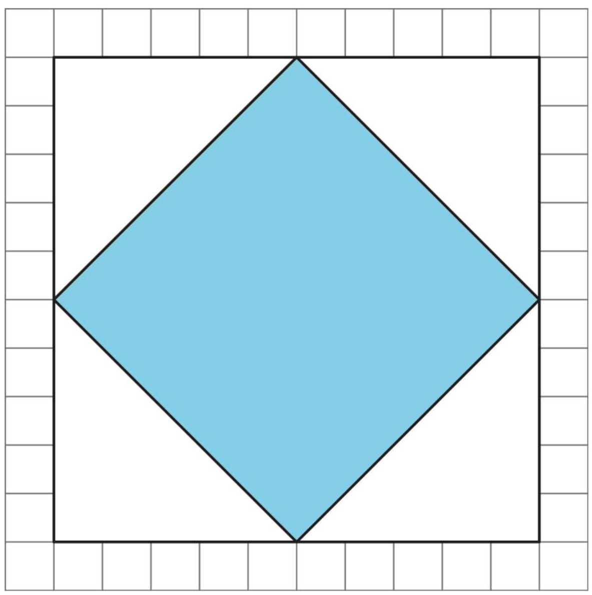
Representation: Internalise Comprehension. Activate or supply background knowledge. Reference examples and displays from previous units of area formulas and methods for finding area.
Supports accessibility for: Memory; Conceptual processing

Student Task Statement

Find the area of each shaded square (in square units).



1.



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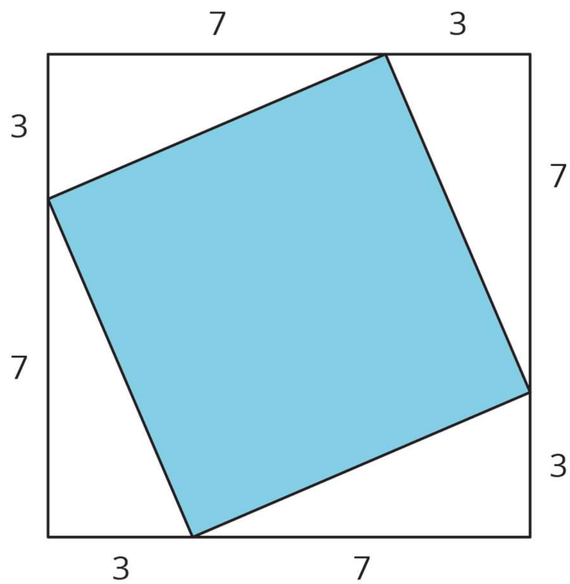


3.

Student Response

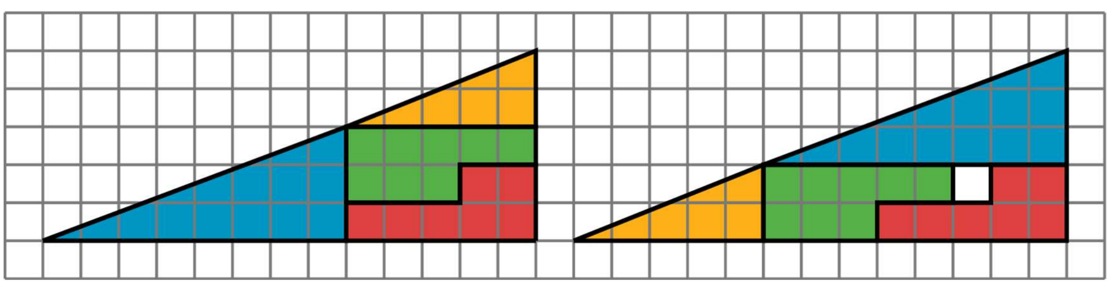
1. 82 square units

2. 50 square units

3. 58 square units

Are You Ready for More?

Any triangle with a base of 13 and a height of 5 has an area of $\frac{65}{2}$.



Both shapes in the figure have been partitioned into the same four pieces. Find the area of each of the pieces, and verify the corresponding parts are the same in each picture. There appears to be one extra square unit of area in the right figure. If all of the pieces have the same area, how is this possible?

Student Response

There is at first an apparent paradox! Each of the two figures is made up of two triangles with areas 12 and 5, and two polygons with areas 8 and 7, for a total of 32. But in the right figure there seems to "magically" be room for an extra square unit of grid space!

The resolution is that, in fact, neither of the two figures are actually triangles! The slopes of the hypotenuses of the two triangles making up the figures are $\frac{2}{5}$ and $\frac{3}{8}$, which are relatively close but not equal, and so do not line up to make one straight hypotenuse for the larger apparent right triangle. So the only flaw in the argument is the expectation that the formula for the area of a triangle should apply.

Activity Synthesis

Select students to explain how they found the area of the third square. Emphasise the strategies of decomposing and either rearranging or finding the areas of the triangles using the triangle area formula.

Speaking, Listening: Compare and Connect. As students prepare a visual display of their work for the last problem, look for students with different methods for calculating the area of the shaded square. As students investigate each other's work, ask them to share what worked well in a particular approach. During this discussion, listen for and amplify the language students use to describe the decomposing, rearranging, and subtracting strategy for finding the area. Then encourage students to make connections between the quantities used to calculate the area of the shaded square and the shapes in the diagram. For example, the quantity 100 represents the area, in square units, of the large square. The quantity 10.5 represents the area, in square units, of one of the triangles. This will foster students' meta-awareness and support constructive conversations as they compare strategies for calculating areas and make connections between quantities and the areas they represent. *Design Principles(s): Cultivate conversation; Maximise meta-awareness*

1.3 Estimating Side Lengths from Areas

10 minutes

In this activity, students use what they know about finding the area of a square from a given side length to think about the converse problem: how do you find the side length of a square with a given area? Students solve this problem in general in the next lesson, but this lesson sets them up for that work.

- First, they work with squares whose side lengths lie along grid lines, so both the side lengths and areas can be found simply by counting length or area units.
- They also see a "tilted" square whose area can be determined by counting square units or by decomposing and rearranging the square or a related figure. In this case, the side lengths are not apparent, so students have to reason that since the area is the same as one of the earlier squares, the side lengths must also be the same. They should check that this appears to be true by tracing one of the squares on tracing paper and seeing that it does, in fact, line up with the other.

The last question asks them to use the insights from the previous questions, tracing paper, or a ruler to estimate the side lengths of some other tilted squares. In the next lesson, they will see other techniques for estimating those side lengths as well.

Monitor for students who use these strategies for the fourth question:

- reason that the side lengths are between 5 and 6 because the areas are between 25 and 36
- measure with tracing paper or a ruler

Instructional Routines

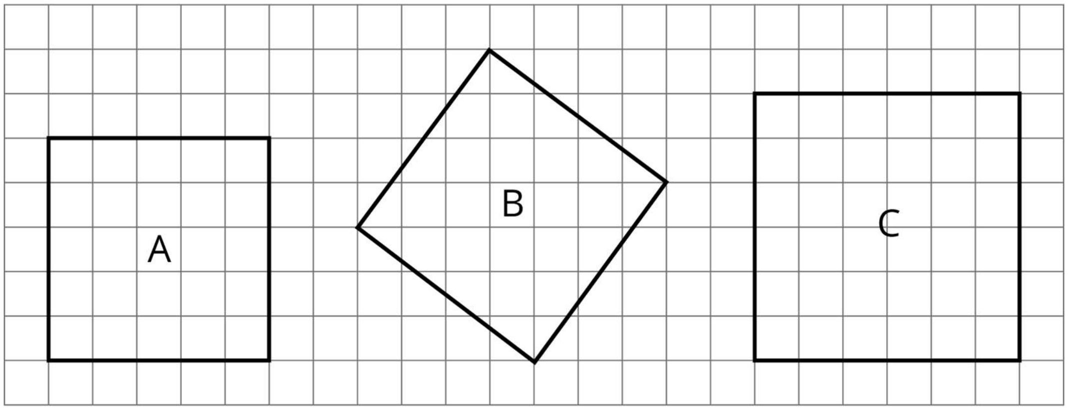
- Anticipate, Monitor, Select, Sequence, Connect
- Co-Craft Questions
- Think Pair Share

Launch

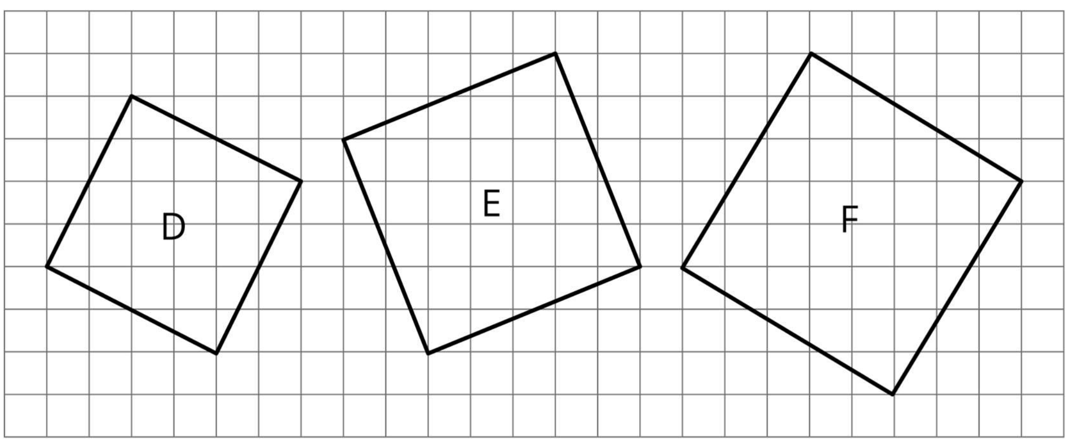
Provide access to tracing paper. Students in groups of 2. 3 minutes of quiet work time followed by a partner discussion on the first three problems. Pause students and make sure everyone agrees on the answers for the first three questions. Students continue with the fourth question, then a whole-class discussion.

Engagement: Develop Effort and Persistence. Encourage and support opportunities for peer interactions. Invite students to talk about their ideas with a partner before writing them down. Display sentence frames to support students when they explain their strategy. For example, "I noticed _____ so I…", "By comparing triangles I…", or "Another strategy could be _____ because…."
Supports accessibility for: Language; Social-emotional skills Conversing, Writing: Co-Craft Questions. Before revealing the questions in this activity, display the diagram of the three squares and invite students to write possible mathematical questions about the diagram. Ask students to compare the questions they generated with a partner before sharing questions with the whole class. Listen for and amplify questions about the side lengths and area of each square. If no student asks about the side lengths or area of each square, ask students to adapt a question to align with the learning goals of this lesson. Then reveal and ask students to work on the actual questions of the task. This routine will help develop students' meta-awareness of language as they generate questions about the side lengths and areas of squares in the coordinate plane.
Design Principle(s): Maximise meta-awareness

Student Task Statement



1. What is the side length of square A? What is its area?
2. What is the side length of square C? What is its area?
3. What is the area of square B? What is its side length? (Use tracing paper to check your answer to this.)
4. Find the areas of squares D, E, and F. Which of these squares must have a side length that is greater than 5 but less than 6? Explain how you know.



Student Response

1. 5 units. 25 square units
2. 6 units. 36 square units
3. 25 square units. 5 units
4. D: 20 square units. E: 29 square units. F: 34 square units. E and F have side lengths between 5 and 6 because their areas are between 25 and 36.

Activity Synthesis

A key take-away from this activity is that if the area of a square is in between the areas of two other squares, then the side lengths are also in between, which in the case of the squares in this task can be reinforced by estimating with tracing paper.

Select previously identified students to share their solutions. Sequence with a reasoning strategy followed by a measuring strategy using a ruler or tracing paper. Use the measuring strategy to verify the reasoning strategy.

1.4 Making Squares

Optional: 15 minutes (there is a digital version of this activity)

In this activity, students use squares with known areas to determine the total area of five shapes. How students determine the area of the shapes is left open-ended on purpose. Students may calculate each shape individually, or, with a bit of rearranging, they may "fit" the shapes into the squares. It is possible to fit all 5 shapes into the two smaller squares or into the one larger square. Identify groups who calculate the area of the shapes individually versus those fitting the shapes into squares.

While these shapes are likely to seem arbitrary to students in this lesson, they are actually part of a transformations-based proof of Pythagoras' theorem that students will revisit in a future lesson. The five cut-out shapes in this activity should be reserved at the end of the activity for future use.

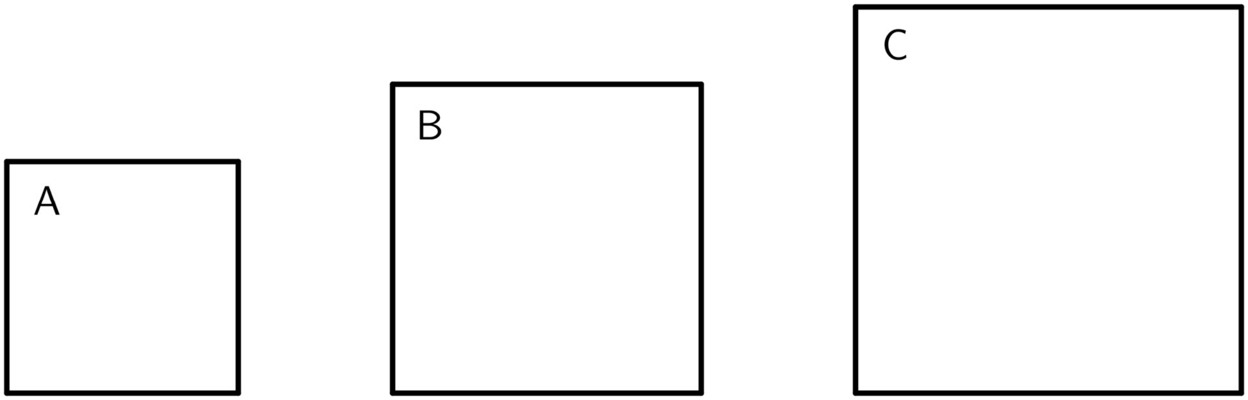
You will need the blackline master for this activity unless you are using the digital version.

Instructional Routines

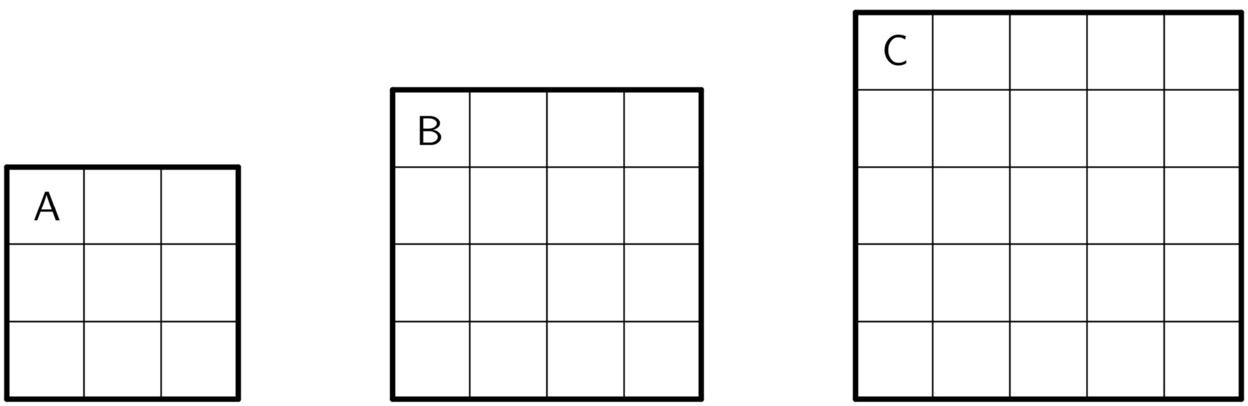
- Discussion Supports

Launch

Display the image of three squares for all to see. Ask students which is larger: the combined area of the two smaller squares A and B or the area of the one larger square C? Give students 30–60 seconds of quiet think time, and then poll the class for their responses. Display a tally of student responses for all to see.



Display the image of three squares with grids for all to see. Tell students that these are the same three squares as before and repeat the previous question. Give students 30–60 seconds of quiet think time, and then select 1–2 students to explain their reasoning. (9 square units and 16 square units is the same as 25 square units, so the combined area of squares A and B is the same as the area of square C.)



Arrange students in groups of 2 and distribute one copy of the three squares half-sheet and five pre-cut shapes from the blackline master to each group. The five pre-cut shapes are labelled on one side to facilitate conversation.

Students using the digital version of the curriculum have an applet to use.

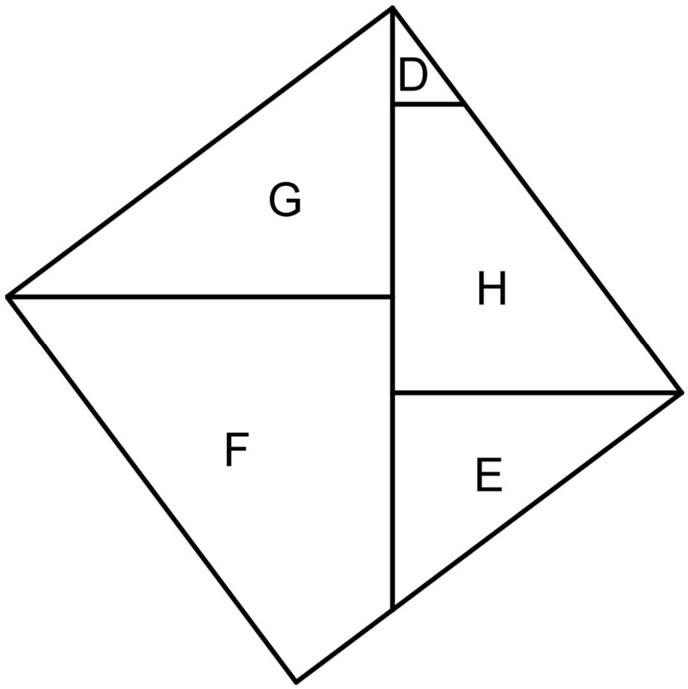
Student Task Statement

Your teacher will give your group a sheet with three squares and 5 cut out shapes labelled D, E, F, G, and H. Use the squares to find the total area of the five shapes. Assume each small square is equal to 1 square unit.

Student Response

The total area of the 5 shapes is 25 square units.

See the blackline master for how the 5 shapes fit into the two smaller squares. Sample response:



Activity Synthesis

The purpose of this discussion is for students to share the different ways they determined the total area of the shapes and to get students thinking about strategies for calculating area and rearranging figures in preparation for future lessons. Select previously identified groups to share, starting with those who calculated each shape individually. If any group fit the shapes into both the two smaller squares and then into the larger square, ask them to demonstrate how they did so for the class.

Engagement: Develop Effort and Persistence. Break the class into small discussion groups and then invite a representative from each group to report back to the whole class. *Supports accessibility for: Attention; Social-emotional skills Speaking, Listening: Discussion Supports.* As students share the total area of the five shapes, press for details in students' reasoning by asking how they know the total area is 25 square units. Listen for and amplify the language students use to describe how they rearranged the five shapes to fit into either the two smaller squares or the larger square. Help students understand that if the shapes fit in a square perfectly without overlapping, then the combined area of the shapes must be the same as the area of the square. This will support rich and inclusive discussion about a strategy for calculating the total area of several shapes by arranging them into a square. *Design Principle(s): Support sense-making*

Lesson Synthesis

The purpose of this synthesis is to reinforce the relationship between the side length of a square and its area.

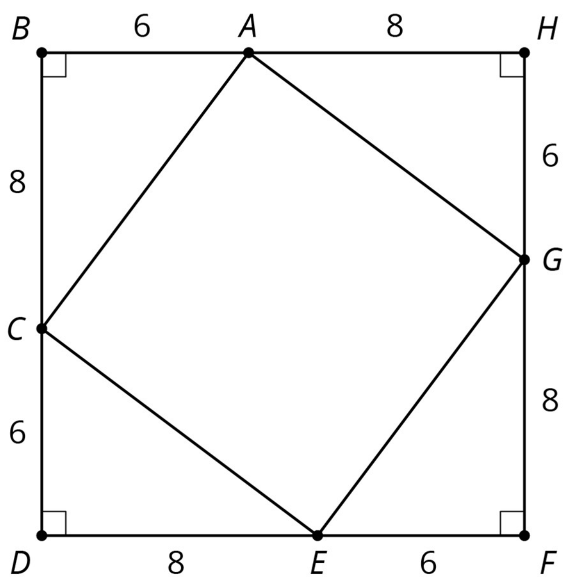
- "If you have a square and know the side length, how can you find the area?" (Multiply the side length by itself; square the side length.)
- "How can you find the area of a square if you don't know the side lengths?" (Enclose it in a square whose side lengths you can find, find the areas of the triangles, and subtract them from the area of the big square.)
- "If you have a square and know the area, how can you find the side length?" (Find the number you multiply by itself/square that equals the area.)

1.5 It's a Square

Cool Down: 5 minutes

Student Task Statement

Find the area and side length of square $ACEG$.



Student Response

$A = 100, s = 10$

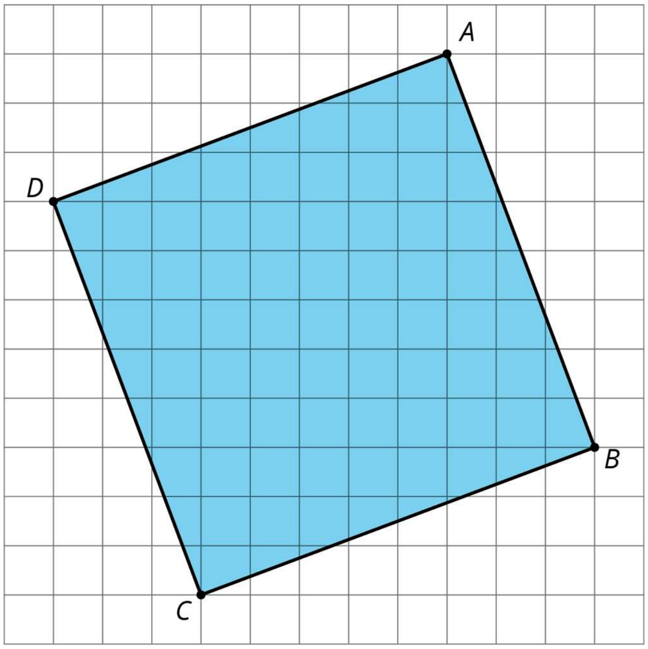
Student Lesson Summary

The area of a square with side length 12 units is 12^2 or 144 units^2.

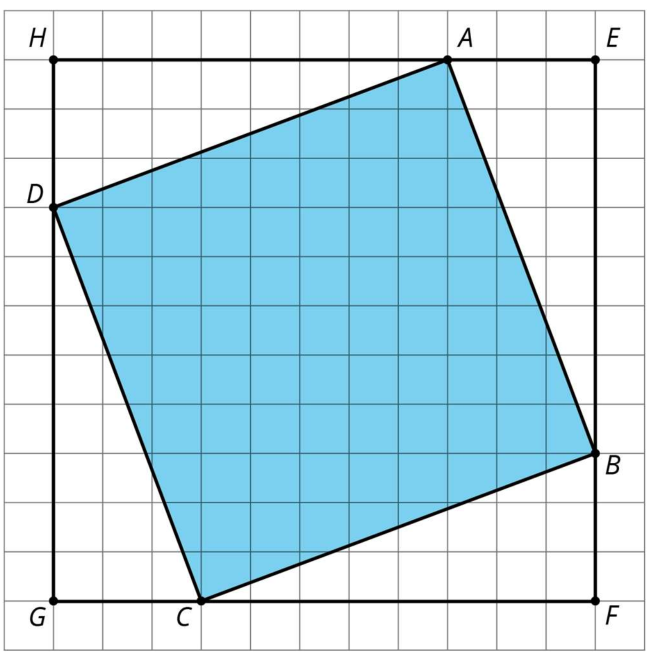
The side length of a square with area 900 units^2 is 30 units because $30^2 = 900$.

Sometimes we want to find the area of a square but we don't know the side length. For example, how can we find the area of square $ABCD$?

One way is to enclose it in a square whose side lengths we do know.



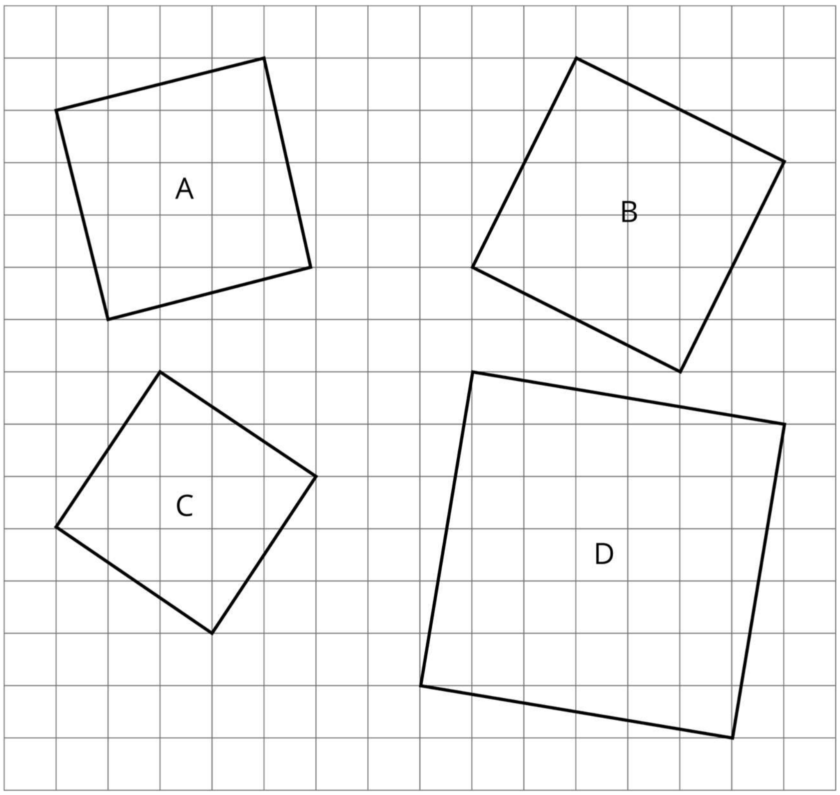
The outside square $EFGH$ has side lengths of 11 units, so its area is 121 units^2. The area of each of the four triangles is $\frac{1}{2} \times 8 \times 3 = 12$, so the area of all four together is $4 \times 12 = 48$ units^2. To get the area of the shaded square, we can take the area of the outside square and subtract the areas of the 4 triangles. So the area of the shaded square $ABCD$ is $121 - 48 = 73$ or 73 units^2.



Lesson 1 Practice Problems

1. **Problem 1 Statement**

   Find the area of each square. Each grid square represents 1 square unit.



Solution

a. 17 square units

b. 20 square units

c. 13 square units

d. 37 square units

2. **Problem 2 Statement**

   Find the length of a side of a square if its area is:

a. 81 square inches

b. $\frac{4}{25}$ cm^2

c. 0.49 square units

d. m^2 square units

Solution

a. 9 inches

b. $\frac{2}{5}$ cm

c. 0.7 units

d. m units

3. **Problem 3 Statement**

Find the area of a square if its side length is:

a. 3 inches

b. 7 units

c. 100 cm

d. 40 inches

e. x units

Solution

a. 9 square inches

b. 49 square units

c. 10 000 cm^2

d. 1 600 square inches

e. x^2 square units

4. **Problem 4 Statement**

Evaluate $(3.1 \times 10^4) \times (2 \times 10^6)$. Choose the correct answer:

a. 5.1×10^{10}

b. 5.1×10^{24}

c. 6.2×10^{10}

d. 6.2×10^{24}

Solution C

5. **Problem 5 Statement**

Noah reads the problem, "Evaluate each expression, giving the answer in standard form." The first problem part is: $5.4 \times 10^5 + 2.3 \times 10^4$.

Noah says, "I can rewrite 5.4×10^5 as 54×10^4. Now I can add the numbers: $54 \times 10^4 + 2.3 \times 10^4 = 56.3 \times 10^4$."

Do you agree with Noah's solution to the problem? Explain your reasoning.

Solution

Answers vary. Sample response: I don't agree with Noah's solution. His calculations are correct, but his final answer is not in standard form. To finish the problem, he should convert his answer to the form 5.63×10^5.

6. **Problem 6 Statement**

Select **all** the expressions that are equivalent to 3^8.

a. $(3^2)^4$

b. 8^3

c. $3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3$

d. $(3^4)^2$

e. $\frac{3^6}{3^{-2}}$

f. $3^6 \times 10^2$

Solution ["A", "C", "D", "E"]