Duals: $\dfrac{1}{1+x^2}$ and $\dfrac{1}{1-x^2}$

Students studying power series are sometimes surprised to discover that the interval of convergence of the power series of $\frac{1}{1+x^2}$ is (-1, 1). To many that appears strange since the function is clearly differentiable for all real *x*. When they study complex analysis they discover that the problem are the poles at $z=\pm i$. While not a proof per se, the phantom graph of $\frac{1}{1+x^2}$ is $y=\frac{1}{1-z^2}$, $x=0$ and this more clearly is limited to $|z|<1$.

It turns out that the phantom graph of $\frac{1}{1-x^2}$ is just $y=\frac{1}{1+z^2}$, $x=0$. Hence, my calling them duals.

Derivation: If $f(x)=\frac{1}{1+x^2}$ then

$$f(u+iv)=\frac{1}{\left(1+(u+iv)^2\right)}=\frac{1}{(1+u^2-v^2)+i2uv}=\frac{1+u^2-v^2-i2uv}{\left(1+u^2-v^2\right)^2+4u^2v^2}$$

The imaginary part is zero if either *v*=0 or *u*=0. Ignoring the first, we find the real part when *u*=0 is $\frac{1-v^2}{\left(1-v^2\right)^2}=\frac{1}{1-v^2}$. The phantom graph follows.

A similar calculation works for the phantom graph of $f(x)=\frac{1}{1-x^2}$.