Mathematics Education Innovation

Lesson 5: Reasoning about equations and bar models (Part 2)

Goals

- Coordinate bar models, equations of the form $p(x + q) = r$, and verbal descriptions of the situations.
- Explain (orally and in writing) how to use a bar model to determine the value of an unknown quantity in an equation of the form $p(x + q) = r$.
- Interpret (in writing) the solution to an equation in the context of the situation it represents.

Learning Targets

- I can draw a bar model to represent a situation where there is more than one copy of the same sum and explain what the parts of the diagram represent.
- I can find a solution to an equation by reasoning about a bar model or about what value would make the equation true.

Lesson Narrative

This lesson parallels the previous one, except the focus is on situations that lead to equations of the form $p(x + q) = r$. Bar models are used to help students understand why these situations can be represented with equations of this form, and to help them reason about solving equations of this form. Students also attend to the meaning of the equation's solution in the context. Note that we are not generalising solution methods yet; just using diagrams as a tool to reason about solving equations.

Addressing

- Solve multi-step real-life and mathematical problems posed with positive and negative rational numbers in any form (whole numbers, fractions, and decimals), using tools strategically. Apply properties of operations to calculate with numbers in any form; convert between forms as appropriate; and assess the reasonableness of answers using mental computation and estimation strategies.
- Use variables to represent quantities in a real-world or mathematical problem, and construct simple equations and inequalities to solve problems by reasoning about the quantities.
- Solve word problems leading to equations of the form $px + q = r$ and $p(x + q) = r$, where p, q, and r are specific rational numbers. Solve equations of these forms fluently. Compare an algebraic solution to an arithmetic solution, identifying the sequence of the operations used in each approach. For example, the perimeter of a rectangle is 54 cm. Its length is 6 cm. What is its width?

Building Towards

- Solve word problems leading to equations of the form $px + q = r$ and $p(x + q) = r$, where p, q, and r are specific rational numbers. Solve equations of these forms fluently. Compare an algebraic solution to an arithmetic solution, identifying the sequence of the operations used in each approach. For example, the perimeter of a rectangle is 54 cm. Its length is 6 cm. What is its width?

Instructional Routines

- Algebra Talk
- Stronger and Clearer Each Time
- Collect and Display
- Discussion Supports

Student Learning Goals

Let's use bar models to help answer questions about situations where the equation has parentheses.

5.1 Algebra Talk: Seeing Structure

Warm Up: 10 minutes

This warm-up parallels the one in the previous lesson. The purpose of this Algebra Talk is to elicit strategies and understandings students have for solving equations. These understandings help students develop fluency and will be helpful later in this unit when students will need to be able to come up with ways to solve equations of this form. While four equations are given, it may not be possible to share every strategy. Consider gathering only two or three different strategies per problem, saving most of the time for the final question.

Students should understand the meaning of *solution to an equation* from earlier work in KS3 as well as from work earlier in this unit, but this is a good opportunity to re-emphasise the idea.

In this string of equations, each equation has the same solution. Digging into why this is the case requires noticing and using the structure of the equations. Noticing and using the structure of an equation is an important part of fluency in solving equations.

Instructional Routines

- Algebra Talk
- Discussion Supports

Launch

Display one equation at a time. Give students 30 seconds of quiet think time for each equation and ask them to give a signal when they have an answer and a strategy. Keep all equations displayed throughout the talk. Follow with a whole-class discussion.

Representation: Internalise Comprehension. To support working memory, provide students with sticky notes or mini whiteboards.
Supports accessibility for: Memory; Organisation

Student Task Statement

Solve each equation mentally.

$x - 1 = 5$

$2(x - 1) = 10$

$3(x - 1) = 15$

$500 = 100(x - 1)$

Student Response

- 6 is the solution because $6 - 1 = 5$.
- 6. Possible strategies: Trial and error to arrive at $2(6 - 1) = 10$, noticing that the equation is 2 times something is 10, so the something must be a 5, applying the distributive property to get $2x - 2 = 10$, and reasoning from there.
- 6

$x = 6$ is a solution to each equation because in each one, $x - 1$ has to equal 5.

Activity Synthesis

This discussion may go quickly, because students are likely to recognise similarities between this equation string and the one in the previous lesson's warm-up.

Ask students to share their strategies for each problem. Record and display their responses for all to see. To involve more students in the conversation, consider asking:

- "Who can restate ___'s reasoning in a different way?"
- "Did anyone have the same strategy but would explain it differently?"
- "Did anyone solve the equation in a different way?"
- "Does anyone want to add on to _____'s strategy?"
- "Do you agree or disagree? Why?"

Speaking: Discussion Supports.: Display sentence frames to support students when they explain their strategy. For example, "First, I _____ because . . ." or "I noticed _____ so I . . . ." Some students may benefit from the opportunity to rehearse what they will say with a partner before they share with the whole class.
Design Principle(s): Optimise output (for explanation)

5.2 More Situations and Diagrams

15 minutes (there is a digital version of this activity)

The purpose of this activity is to work toward showing students that some situations can be represented by an equation of the form $p(x + q) = r$ (or equivalent). In this activity, students are simply tasked with drawing a bar model to represent each situation. In the following activity, they will work with corresponding equations.

For each question, monitor for one student with a correct diagram. Press students to explain what any variables used to label the diagram represent in the situation.

Instructional Routines

- Stronger and Clearer Each Time

Launch

Ensure students understand that the work of this task is to draw a bar model to represent each situation. There is no requirement to write an equation or solve a problem yet.

Arrange students in groups of 2. Give 5–10 minutes to work individually or with their partner, followed by a whole-class discussion.

Action and Expression: Develop Expression and Communication. Maintain a display of important terms, vocabulary, and examples. During the launch, take time to review examples of drawing a bar model based on situations from previous lessons that students will need to access for this activity. Consider providing step-by-step directions that generalise the process using student input and ideas.
Supports accessibility for: Memory; Language

Student Task Statement

Draw a bar model to represent each situation. For some of the situations, you need to decide what to represent with a variable.

1. Each of 5 gift bags contains x pencils. Tyler adds 3 more pencils to each bag. Altogether, the gift bags contain 20 pencils.

2. Noah drew an equilateral triangle with sides of length 5 inches. He wants to increase the length of each side by x inches so the triangle is still equilateral and has a perimeter of 20 inches.

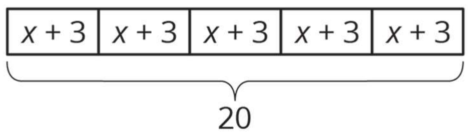
3. An art class charges each student £3 to attend plus a fee for supplies. Today, £20 was collected for the 5 students attending the class.

4. Elena ran 20 miles this week, which was three times as far as Clare ran this week. Clare ran 5 more miles this week than she did last week.

Student Response

Answers vary. Sample diagrams:

A



B

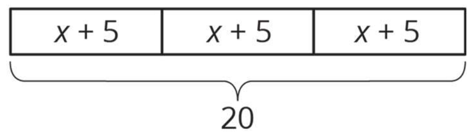


Diagram A corresponds to situations 1 and 3. Diagram B corresponds to situations 2 and 4.

Activity Synthesis

Select one student for each situation to present their correct diagram. Ensure that students explain the meaning of any variables used to label their diagram. Possible questions for discussion:

- "For the situations with no x, how did you decide what quantity to represent with a variable?" (Think about which amount is unknown but has a relationship to one or more other amounts in the story.)
- "What does the variable you used to label the diagram represent in the story?"
- "Did any situations have the same diagrams? How can you tell from the story that the diagrams would be the same?" (Same number of equal parts, same amount for the total.)

Speaking, Representing, Reading: Stronger and Clearer Each Time. Ask students to explain to a partner how they created the bar model to represent the situation "An art class charges each student £3 to attend plus a fee for supplies. Today, £20 was collected for the 5 students attending the class." Ask listeners to press for details in the arrangement of the grouped quantities (e.g., "Explain how you chose what values go in each box."). When roles are switched, listeners can press for details in what "x" represents in the diagram. Allow students to revise their diagrams, if necessary, based on the feedback they received from their partner. Once their revision is complete, invite students to turn to a new partner to explain their revised diagram. This will help students productively engage in discussion as they make connections between written situations and visual diagrams.
Design Principle(s): Optimise output (for explanation); Cultivate conversation

5.3 More Situations, Diagrams, and Equations

10 minutes

This activity is a continuation of the previous one. Students match each situation from the previous activity with an equation, solve the equation by any method that makes sense to them, and interpret the meaning of the solution. Students are still using any method that makes sense to them to reason about a solution. In later lessons, a balance diagram representation will be used to justify more efficient methods for solving.

For each equation, monitor for a student using their diagram to reason about the solution and a student using the structure of the equation to reason about the solution.

Instructional Routines

- Collect and Display

Launch

Keep students in the same groups. 5 minutes to work individually or with a partner, followed by a whole-class discussion.

Engagement: Develop Effort and Persistence. Encourage and support opportunities for peer interactions. Prior to the whole-class discussion, invite students to share their work with a partner. Display sentence frames to support student conversation such as "To find the solution, first, I _____ because...", "I made this match because I noticed...", "Why did you...?", or "I agree/disagree because…"
Supports accessibility for: Language; Social-emotional skills Speaking, Representing: Collect and Display. As students share their ideas about how the equations match the situations, listen for and collect students' description of the situation (e.g., "5 gift bags, x pencils, adds 3 more, 20 pencils") with the corresponding equation. Remind students to borrow language from the displayed examples while describing what each solution tells about the situation, after the matching is complete. This will help students make connection between language, diagrams, and equations.
Design Principle(s): Support sense-making; Maximise meta-awareness

Student Task Statement

Each situation in the previous activity is represented by one of the equations.

- $(x+3)\times 5=20$
- $3(x+5)=20$

1. Match each situation to an equation.
2. Find the solution to each equation. Use your diagrams to help you reason.
3. What does each solution tell you about its situation?

Student Response

- $(x + 3) \times 5 = 20$: Situations 1 (gift bags) and 3 (art class)
- $3(x + 5) = 20$: Situations 2 (triangle perimeter) and 4 (miles run)
- $(x + 3) \times 5 = 20$ $\quad x = 1$
- $3(x + 5) = 20$ $\quad x = 1\frac{2}{3}$

a. There was originally one pencil in each bag.

b. Noah increased the length of each side by $1\frac{2}{3}$ inches.

c. The fee for supplies is £1.

d. Clare ran $1\frac{2}{3}$ miles last week.

Are You Ready for More?

Han, his sister, his dad, and his grandmother step onto a crowded bus with only 3 open seats for a 42-minute ride. They decide Han's grandmother should sit for the entire ride. Han, his sister, and his dad take turns sitting in the remaining two seats, and Han's dad sits 1.5 times as long as both Han and his sister. How many minutes did each one spend sitting?

Student Response

Han's grandmother: 42, Han's dad: 36, Han: 24, Han's sister: 24

Activity Synthesis

For each equation, ask one student who reasoned with the diagram and one who reasoned only about the equation to explain their solutions. Display the diagram and the equation side by side, drawing connections between the two representations. If no students bring up one or both of these approaches, demonstrate the manoeuvres on a diagram side by side with the manoeuvres on the corresponding equation. For example, "I divided the number of gift bags by 5, leaving me with 4 pencils per gift bag. Since Tyler added 3 pencils to each gift bag, there must have been 1 pencil in each gift bag to start," can be shown on a bar model and on a corresponding equation. It is not necessary to invoke the more abstract language of "doing the same thing to each side" of an equation yet.

Lesson Synthesis

Display one of the situations from the lesson and its corresponding equation. Ask students to explain:

- "What does each number and letter in the equation represent in the situation?"
- "What is the reason for each operation (multiplication or addition) used in the equation?"

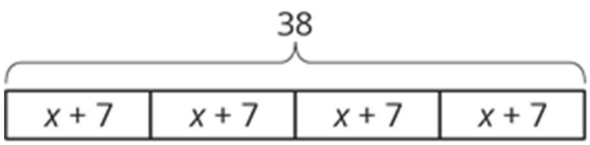
- "What is the solution to the equation? What does it mean to be a solution to an equation? What does the solution represent in the situation?"

5.4 More Finding Solutions

Cool Down: 5 minutes

Student Task Statement

Here is a diagram and its corresponding equation. Find the solution to the equation and explain your reasoning.



$4(x + 7) = 38$

Student Response

$x = 2\frac{1}{2}$.

Sample reasoning: the bar model is in 4 equal pieces, each of which represents $\frac{38}{4}$ $\left(\text{or } 9\frac{1}{2}\right)$. $x + 7 = 9\frac{1}{2}$, so x must be $2\frac{1}{2}$.

Student Lesson Summary

Equations with brackets can represent a variety of situations.

1. Lin volunteers at a hospital and is preparing toy baskets for children who are patients. She adds 2 items to each basket, after which the supervisor's list shows that 140 toys have been packed into a group of 10 baskets. Lin wants to know how many toys were in each basket before she added the items.

2. A large store has the same number of workers on each of 2 teams to handle different shifts. They decide to add 10 workers to each team, bringing the total number of workers to 140. An executive at the company that runs this chain of stores wants to know how many employees were in each team before the increase.

Each bag in the first story has an unknown number of toys, x, that is increased by 2. Then ten groups of $x + 2$ give a total of 140 toys. An equation representing this situation is $10(x + 2) = 140$. Since 10 times a number is 140, that number is 14, which is the total number of items in each bag. Before Lin added the 2 items there were $14 - 2$ or 12 toys in each bag.

The executive in the second story knows that the size of each team of y employees has been increased by 10. There are now 2 teams of $y + 10$ each. An equation representing this situation is $2(y + 10) = 140$. Since 2 times an amount is 140, that amount is 70, which is the new size of each team. The value of y is $70 - 10$ or 60. There were 60 employees on each team before the increase.

Lesson 5 Practice Problems

1. **Problem 1 Statement**

   Here are some prices customers paid for different items at a farmer's market. Find the cost for 1 pound of each item.

   a. £5 for 4 pounds of apples

   b. £3.50 for $\frac{1}{2}$ pound of cheese

   c. £8.25 for $1\frac{1}{2}$ pounds of coffee beans

   d. £6.75 for $\frac{3}{4}$ pounds of fudge

   e. £5.50 for a $6\frac{1}{4}$ pound pumpkin

   Solution

   a. £1.25

   b. £7

   c. £5.50

   d. £9

   e. £0.88

2. **Problem 2 Statement**

   Find the products.

   a. $\frac{2}{3} \times \left(\frac{-4}{5}\right)$

   b. $\left(\frac{-5}{7}\right) \times \left(\frac{7}{5}\right)$

   c. $\left(\frac{-2}{39}\right) \times 39$

   d. $\left(\frac{2}{5}\right) \times \left(\frac{-3}{4}\right)$

Solution

a. $\frac{-8}{15}$

b. -1

c. -2

d. $\frac{-3}{10}$ or equivalent

3. **Problem 3 Statement**

Here are two stories:

– A family buys 6 tickets to a show. They also *each* spend £3 on a snack. They spend £24 on the show.

– Diego has 24 ounces of juice. He pours equal amounts for each of his 3 friends, and then adds 6 more ounces for each.

Here are two equations:

– $3(x + 6) = 24$

– $6(x + 3) = 24$

a. Which equation represents which story?

b. What does x represent in each equation?

c. Find the solution to each equation. Explain or show your reasoning.

d. What does each solution tell you about its situation?

Solution

a. Family at the show: $6(x + 3) = 24$, Diego's juice: $3(x + 6) = 24$

b. Family at the show: x represents the cost of a ticket. Diego's juice: x represents the number of ounces of juice Diego originally poured for each friend.

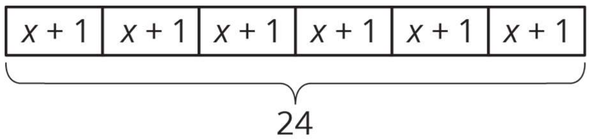
c. $6(x + 3) = 24 \quad x = 1$

$3(x + 6) = 24 \quad x = 2$

d. Tickets to the show cost £1. Diego originally poured 2 ounces of juice.

4. **Problem 4 Statement**

Here is a diagram and its corresponding equation. Find the solution to the equation and explain your reasoning.



$6(x+1) = 24$

Solution

3. Sample response: in the bar model, there are six units of $x+1$ that make 24, so $x+1$ must be $24 \div 6$, which is 4. Since $x+1=4$, $x=3$.

5. **Problem 5 Statement**

Below is a set of data about temperatures. The *range* of a set of data is the distance between the lowest and highest value in the set. What is the range of these temperatures?

9°C,-3°C, 22°C,-5°C, 11°C, 15°C

Solution

27

6. **Problem 6 Statement**

A store is having a 25% off sale on all shirts. Show two different ways to calculate the sale price for a shirt that normally costs £24.

Solution

Answers vary. Possible strategies:

- $(0.25) \times 24 = 6$, and $24 - 6 = 18$ (find 25% of £24 and subtract this from £24)
- $1 - 0.25 = 0.75$, and $(0.75) \times 24 = 18$ (find 75% of £24)
- $24 \div 4 = 6$, and $3 \times 6 = 18$ (find 25% of £24 and multiply this by 3)