AREA OF PARABOLIC SEGMENT

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Introduction

Archimedes stated a Proposition in his work "Quadrature of the Parabola"

> *The area of any segment of a parabola is four-thirds of [the area of] the triangle which has the same base as the segment and equal height.*

In this proposition he defines the *base* as the straight line (the chord) which intersects the parabola and the *height* as "the greatest perpendicular drawn from the curve to the base of the segment." We will return to this below. Our objective is to prove the above proposition, using modern mathematics.

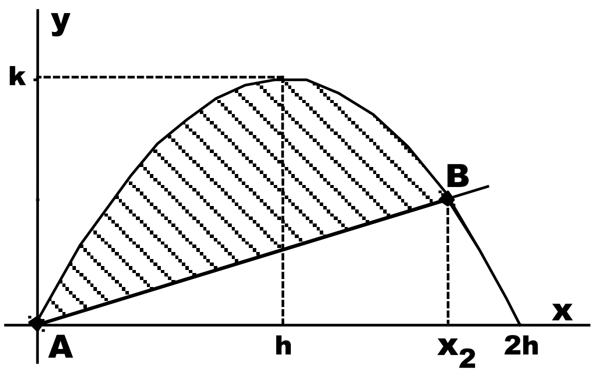

Figure 1

First we need a sketch of the figure (the parabola and the chord). It is essential to recognize that the orientation of the figure is arbitrary, since the figure and its enclosed area are invariant under translation and/or rotation. Thus we might as well position the figure at a convenient location, hoping to simplify the math. In Figure 1 we have a sketch of the parabola and the line. The area of interest is that between the parabola and the line AB. We have oriented the figure so that the left intersection point A occurs at the origin. The equation of the line is just

$$y_L(x) = mx$$

since the y-intercept b is zero, while that of the parabola is, in general vertex form with (h,k) the x and y coordinates of the vertex,

$$y_P(x) = a(x-h)^2 + k$$

In this case the parabola passes through the origin, so that

$$0 = a(0-h)^2 + k \qquad a = \frac{-k}{h^2}$$

We will restrict the parameter k to be positive, while the slope m of the chord can be positive, negative, or zero. Parabolas with negative k can be considered a rotation of the one we are using. The next thing we need is the x-coordinate of the point B, since this will be the upper limit of the integral we will be evaluating, for the area between the parabola and the line AB. At point B the ordinate of the line and of the parabola are equal, so that, with x_2 the x-coordinate in question,

$$\frac{-k}{h^2}\left(x_2 - h\right)^2 + k = mx_2$$

$$\boxed{x_2 = h\left(2 - \frac{mh}{k}\right)}$$

Note that, if the slope m is zero so that the intersecting line AB is the x-axis, then x_2 is just $2h$. Then we can consider the offset

$$\Delta x_2 = \frac{-mh^2}{k}$$

of the intersection point B from where it would have been for a horizontal line.

Parabolic Segment Area

Now we are ready to write the integral for the area between the parabola and line. The format is as usual

$$A = \int_{x_1}^{x_2} (\,f(x) - g(x))\,dx$$

with *f(x)* having the larger values, so that the area is positive. In the present case we can write the parabola area as

$$A_p = \int_0^{x_2} \left[a(x-h)^2 + k - mx\right] dx$$

Looking at Fig. 1 we can see that the area between the parabola and the line AB equals the area under the parabola, to the x-axis, minus the area of the triangle ABx_2. The integral above reflects this, since

$$A_p = \int_0^{x_2} \left[a(x-h)^2 + k\right] dx - \frac{1}{2} m x_2^{\,2}$$

with the last term being the area of the triangle. Evaluating the integral gives

$$\int_0^{x_2} \left[a(x-h)^2 + k\right] dx \qquad \frac{a}{3}\left[\left(x_2 - h\right)^3 + h^3\right] + k x_2$$

Then the area is

$$\frac{a}{3}\left[\left(x_2 - h\right)^3 + h^3\right] + k x_2 - \frac{1}{2} m x_2^{\,2}$$

and substituting for a and x_2 we have

$$\frac{-1}{3}\,\frac{k}{h^2}\left[\left[h\left(2 - m\frac{h}{k}\right) - h\right]^3 + h^3\right] + kh\left(2 - m\frac{h}{k}\right) - \frac{1}{2} m h^2 \left(2 - m\frac{h}{k}\right)^2$$

which simplifies to

$$A_p = \frac{h}{6k^2}(2k - mh)^3$$

Inscribed Triangle

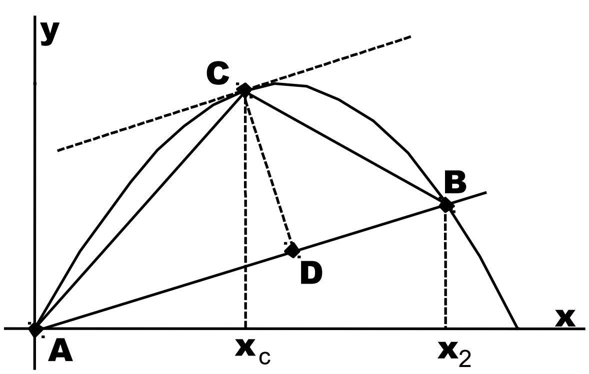

Figure 2

Now we need to define the triangle that lies inside the parabolic segment. We have two of the three points, A and B. The question is, where is the third point, C? Consider Fig. 2. According to Archimedes, the height we want is the longest length for line segment CD, or, equivalently, the maximum vertical distance from the line AB to the parabola, measured along a line perpendicular to the x-axis and passing through the point C. Archimedes uses the term "perpendicular" but he does not define perpendicular to *what.* In any case, we can find the longest distance from the line to the parabola in four ways. They will all produce the same result. We now consider each of these methods for finding the third point of the inscribed triangle.

Point C, Method 1

In the first method we take the derivative of the parabola, and find the point where this slope is equal to the slope of the chord (Fig. 2). In Method 3 below we will see why this yields the correct position for C. We have

$$\frac{d}{dx} y_p(x) = \frac{-2k}{h^2}(x - h)$$

Then if the slopes are equal

$$\frac{-2k}{h^2}\left(x_c - h\right) = m$$

$$x_c = h\left(1 - \frac{mh}{2k}\right)$$

$$y_c = k - \frac{m^2 h^2}{4k}$$

and note that the x-coordinate is just

$$x_c = \frac{x_2}{2}$$

As before with x_2, for x_C we have an offset from the position (h) it would have for zero slope:

$$\Delta x_c = \frac{-mh^2}{2k}$$

We can also observe that the position C we have found leads to the bisection of the line AB. We have that the point bisects the line (x-axis) from 0 to x_2, but it can be shown (by algebra, or, more easily, similar triangles) that this x-coordinate also bisects AB. Archimedes used this fact, but we will not.

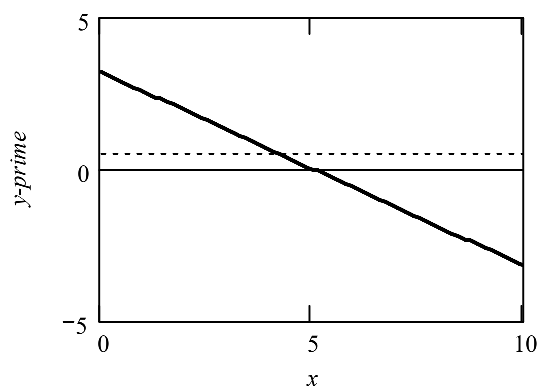

Figure 3

In Fig. 3 above we have a plot of the derivative (slope) of the parabola, with the dotted line representing the value of m. Note that the derivative is zero at the vertex, at x=5.

Point C, Method 2

Another method to find the maximum height, and thus the location of C, is to find the area of the triangle, with the point C left as an unknown. Then we can maximize this area, via a derivative, and find the x-coordinate of C. It can be shown that, in general, with the three points at the vertices of the triangle given, the area is

$$A_T = \frac{1}{2}\left|\begin{pmatrix} x1 & y1 & 1 \\ x2 & y2 & 1 \\ x3 & y3 & 1 \end{pmatrix}\right|$$

For our case, we have

$$A_T = \frac{1}{2}\left|\begin{pmatrix} 0 & 0 & 1 \\ x_2 & mx_2 & 1 \\ x_c & y_c & 1 \end{pmatrix}\right|$$

which is

$$\frac{1}{2}\left[0\left|\begin{pmatrix} mx_2 & 1 \\ y_c & 1 \end{pmatrix}\right| - x_2\left|\begin{pmatrix} 0 & 1 \\ y_c & 1 \end{pmatrix}\right| + x_c\left|\begin{pmatrix} 0 & 1 \\ mx_2 & 1 \end{pmatrix}\right|\right]$$

or

$$\frac{1}{2}x_2\left(y_c - mx_c\right)$$

Using the parabola's equation to find the y-coordinate at C, in terms of x_C, this becomes

$$A_T = \frac{1}{2}x_2\left[a\left(x_c - h\right)^2 + k - mx_c\right]$$

All we need to do with this is differentiate with respect to the free variable x_C, which gives

$$\frac{d}{dx_c}A_T = ax_2\left(x_c - h\right) - \frac{mx_2}{2}$$

and then we solve for x_C when the derivative is zero. This results in

$$x_c = h + \frac{m}{2a} \qquad x_c = h\left(1 - \frac{mh}{2k}\right)$$

as we had before. It would be wise to verify that this is a maximum rather than a minimum, so we take the second derivative of the area,

$$\frac{d^2}{dx^2}A_T = ax_2 = \frac{-k}{h^2}x_2$$

which is negative, since x_2 and k are both positive. Thus the critical point is a maximum.

Point C, Method 3

The third method for finding C is the simplest and most intuitive. We simply maximize the distance δ, measured vertically, from the line AB to the parabola. See Fig. 4. This distance is just

$$\delta(x) = y_P(x) - y_L(x) = \frac{-k}{h^2}(x - h)^2 + k - mx$$

Then

$$\frac{d}{dx}\delta = \frac{d}{dx}y_P - \frac{d}{dx}y_L = \left[\frac{-2k}{h^2}(x - h) - m\right]$$

Note that this is the difference between the slope of the parabola and that of the line, at any x. When this difference of slopes (derivatives) is zero, we have a critical point (max or min) of the distance between the curve and the line. Thus when, for some x, the local slope of the parabola equals the slope of the line, the distance between the parabola and the line is maximized at that value of x. This proves that Method 1 produces the correct x-coordinate.

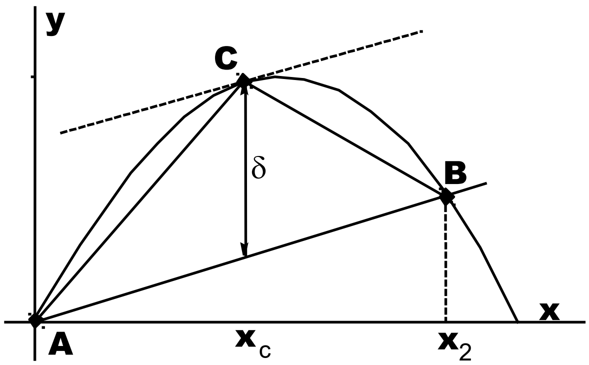

Figure 4

If we set the derivative of the distance to zero, and solve for x, we obtain, as before,

$$\frac{d}{dx}\delta = 0 = \frac{-2k}{h^2}\left(x_c - h\right) - m \qquad x_c = h\left(1 - \frac{mh}{2k}\right)$$

Again we check the second derivative to make sure this is a maximum:

$$\frac{d^2}{dx^2}\delta = \frac{-2k}{h^2}$$

which is negative, so we do have a maximum distance.

Point C, Method 4

A fourth method of finding the triangle's third point is to use the vertical distance again:

$$\delta(x) = y_P(x) - y_L(x) = \frac{-k}{h^2}(x-h)^2 + k - mx$$

but this time we simply recognize that this is the equation for another parabola. Expanding, collecting, completing the square, and re-writing in the vertex form, we obtain

$$\delta(x) = \frac{-k}{h^2}\left[x - \frac{1}{2}(2k - mh)\frac{h}{k}\right]^2 + \frac{1}{4}\frac{(2k-mh)^2}{k}$$

The x-coordinate of the vertex will be the x-coordinate of the maximum of this shifted parabola, and so

$$h_\delta = \frac{h}{2k}(2k - mh) = h\left(1 - \frac{mh}{2k}\right) = x_c$$

These parabolas are illustrated in Fig. 5. Note that the amount of the shift of the vertex is

$$\Delta x = \frac{-mh^2}{2k}$$

compared to the original vertex location at h. Of course, the y-coordinate at the vertex of the shifted parabola is also the maximum distance from the original parabola to the line AB.

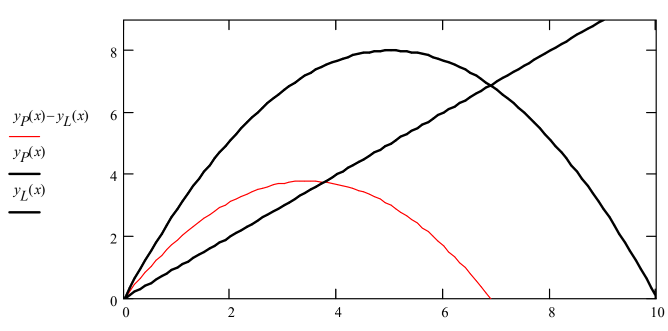

Figure 5

Area Of Triangle

The next step in the analysis is to find the area of the triangle, now that we have defined all three of its vertex points. We will find the area by two methods.

Triangle Area, Method 1

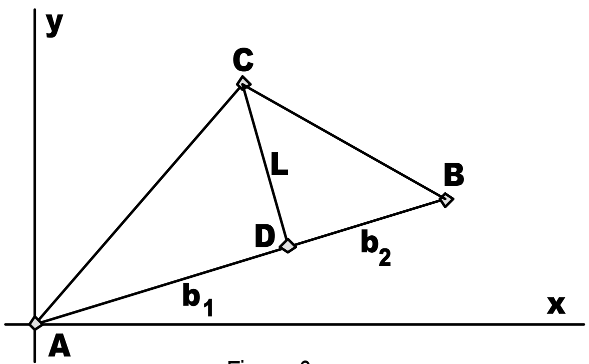

Figure 6

For this approach we will split the triangle into two right triangles. Consider Fig. 6. The line segment AD is the base of the left triangle, and the segment DB is the base for the other triangle. We will need to find the height of both triangles, i.e., the length *L* of segment CD. The area of triangle ABC is

$$A_T = A_1 + A_2 = \frac{1}{2}b_1L + \frac{1}{2}b_2L = \frac{1}{2}bL$$

where *b* is the total length of the base, namely, line segment AB. The length of AB is just a simple Pythagorean relation, since we know the coordinates of points A and B, so that

$$b = \sqrt{x_2^{\,2} + \left(mx_2\right)^2} = x_2\sqrt{1 + m^2}$$

The same result obtains if we use the distance formula, of course. Now we need the height of the triangles, *L*. For this we construct the normal to the tangent line at point C, and recall that the slope of this tangent is *m*. See Fig. 7. Then the point-slope form of the equation of a line yields

$$\frac{y - y_c}{x - x_c} = \frac{-1}{m} \qquad y(x) = \frac{-1}{m}\left(x - x_c\right) + y_c$$

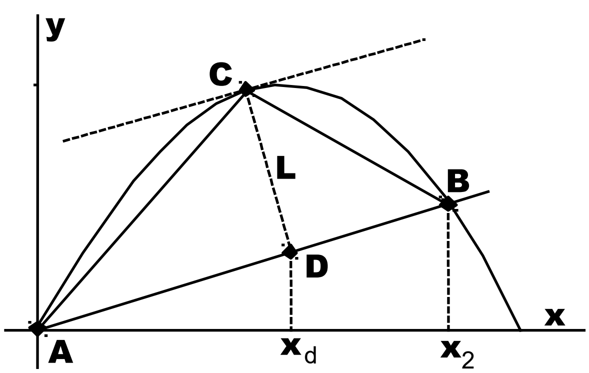

Figure 7

using the fact that the slope of a line normal to another line is the negative reciprocal of the slope of the latter. With this equation we can now find the point of intersection with the base, line AB, whose equation we already know. Setting the y-coordinates equal at the point of intersection D, we have

$$mx_d = y_c - \frac{1}{m}\left(x_d - x_c\right)$$

Solving this for the coordinates of point D,

$$x_d = \frac{x_c + my_c}{1 + m^2} \qquad y_d = mx_d$$

Now we can find the height *L* using the distance formula, since we have the coordinates of the endpoints, C and D. This yields

$$L = \sqrt{\left(x_c - x_d\right)^2 + \left(y_c - y_d\right)^2}$$

Substituting the appropriate expressions for these four variables, and doing a lot of messy algebra, we find

$$L = \frac{1}{4}\frac{(2k - mh)^2}{k\sqrt{1 + m^2}}$$

The area of the triangle will then be, using the result for *b*, above,

$$A_T = \frac{1}{2}\left(x_2\sqrt{1+m^2}\right)\left[\frac{1}{4}\frac{(2k-mh)^2}{k\sqrt{1+m^2}}\right]$$

and, substituting for x_2 we finally get the triangle area

$$A_T = \frac{h}{8k^2}(2k-mh)^3$$

Triangle Area, Method 2

For this method we again use the determinant, for three given points (the vertices) of the triangle. Using the values of the coordinates for point C, we have

$$A_T = \frac{1}{2}\left|\begin{bmatrix} 0 & 0 & 1 \\ x_2 & mx_2 & 1 \\ \frac{x_2}{2} & a\left(\frac{x_2}{2}-h\right)^2+k & 1 \end{bmatrix}\right|$$

Expanding this as we did above,

$$A_T = \frac{1}{2}\left[0\left|\begin{bmatrix} mx_2 & 1 \\ a\left(\frac{x_2}{2}-h\right)^2+k & 1 \end{bmatrix}\right| - x_2\left|\begin{bmatrix} 0 & 1 \\ a\left(\frac{x_2}{2}-h\right)^2+k & 1 \end{bmatrix}\right| + \frac{x_2}{2}\left|\begin{pmatrix} 0 & 1 \\ mx_2 & 1 \end{pmatrix}\right|\right]$$

which is

$$\frac{1}{8}x_2^2\frac{\left(4kh - x_2k - 2mh^2\right)}{h^2}$$

or

$$\frac{1}{2}\left[-x_2\left[-\left[a\left(\frac{x_2}{2}-h\right)^2+k\right]\right]-\frac{mx_2^2}{2}\right]$$

Substitute for x_2 and simplify to obtain the triangle area, which is the same as before:

$$A_T = \frac{h}{8k^2}(2k-mh)^3$$

Ratio Of Areas

Finally, we can find the ratio of the area of the parabolic segment to that of the inscribed triangle:

$$\frac{\frac{h}{6k^2}(2k-mh)^3}{\frac{h}{8k^2}(2k-mh)^3} = \frac{4}{3}$$

just as Archimedes said. Note the simple case when the slope of the chord is zero (horizontal):

$$A_P = \frac{h}{6k^2}(2k)^3 = \frac{4}{3}hk$$

$$A_T = \frac{h}{8k^2}(2k)^3 = hk$$

Monte Carlo Experiment

These calculations were checked via Monte Carlo integration, to find the areas of the parabolic segment, and the triangle, and their ratio. This provided a successful independent check of the calculations above.

Parabolic-Segment Sampling

The sampling for the parabolic segment is done as follows. (1) Sample a uniform random number between 0 and x_2. (2) Sample another, independent uniform random number between the minimum for y and k (the maximum). This sampling is thus of a rectangular region which bounds the parabolic segment (see Fig. 8). The minimum of y will be either 0, if m is positive, or $y(x_2)$ if m is negative. In the figure, the solid rectangle represents the area sampled for positive m, and the dotted rectangle is the region sampled for negative m. Note that if m is negative, x_2 is larger than $2h$, the x-intercept of the parabola.

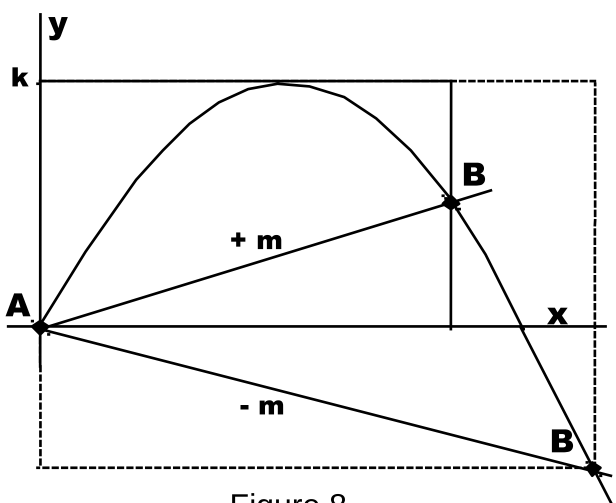

Figure 8

(3) Then, for each paired sample of an x and y, if

$$y_{sample} \le \frac{-k}{h^2}\left(x_{sample} - h\right)^2 + k$$

AND

$$y_{sample} \ge mx_{sample}$$

then the sample fell inside the parabolic segment, and it is scored as a hit. This process is repeated a large number of times, here, 1000. Then the ratio of the number of hits to the number of trials (1000) multiplied by the area of the sampling rectangle can be shown to be an estimate of the area of the region of interest inside that rectangle.

Triangle Sampling

The process for the triangle is similar, except that the conditional tests are a bit more complicated. We need the sample pair to fall inside the region defined by three line segments. Also, in this case, the sampling rectangle has its upper bound at the y-value of point C rather than at k. The minimum is defined as before. Then a hit is scored if

$$y_{sample} \ge mx_{sample} \quad \text{AND} \quad y_{sample} \le \frac{y_c}{x_c}x_{sample}$$

AND

$$y_{sample} \le mx_2 + \frac{y_c - mx_2}{x_c - x_2}\left(x_{sample} - x_2\right)$$

where these expressions define the line segments for the inscribed triangle (AB, AC, CB, respectively). The area estimate is obtained in the same manner as for the parabolic segment.