Mathematics Education Innovation

Lesson 15: Part-part-whole ratios

Goals

- Comprehend the word "parts" as an unspecified unit in sentences (written and spoken) describing ratios.
- Draw and label a bar model to solve problems involving ratios and the total amount. Explain (orally) the solution method.

Learning Targets

- I can create bar models to help me reason about problems involving a ratio and a total amount.
- I can solve problems when I know a ratio and a total amount.

Lesson Narrative

Up to this point, students have worked with ratios of quantities where the units are the same (e.g., cups to cups) and ratios of quantities where the units are different (e.g., miles to hours). Sometimes in the first case, the sum of the quantities makes sense in the context, and we can ask questions about the total amount as well as the component parts. For example, when mixing 3 cups of yellow paint with 2 cups of blue paint, we get a total of 5 cups of green paint. (Notice that this does not always work; 3 cups of water mixed with 2 cups of dry oatmeal will not make 5 cups of soggy oatmeal.) In the paint scenario, the ratio of yellow paint to blue paint to green paint is $3 : 2 : 5$. Furthermore, if we double the amount of both yellow and blue paint, we will double the amount of green paint. In general, if the ratio of yellow to blue paint is equivalent, the ratio of yellow to blue to green paint will also be the equivalent. We can see this is always true because of the distributive property:

$a : b : (a + b)$ is equivalent to $2a : 2b : (2a + 2b)$ because $2a + 2b = 2(a + b)$.

These ratios are sometimes called "part-part-whole" ratios.

In this lesson, students learn about **bar models** as a handy tool to represent ratios with the same units and as a way to reason about individual quantities (the parts) and the total quantity (the whole). Here students also see ratios expressed not in terms of specific units (millilitres, cups, square feet, etc.) but in terms of "parts" (e.g., the recipe calls for 2 parts of glue to 1 part of water).

Building On

- Apply properties of operations as strategies to multiply and divide. Students need not use formal terms for these properties. Examples: If $6 \times 4 = 24$ is known, then $4 \times 6 = 24$ is also known. (Commutative property of multiplication.) $3 \times 5 \times 2$ can be found by $3 \times 5 = 15$, then $15 \times 2 = 30$, or by $5 \times 2 = 10$, then $3 \times 10 = 30$. (Associative property of multiplication.) Knowing that $8 \times 5 = 40$ and $8 \times 2 = 16$, one can find 8×7 as $8 \times (5 + 2) = (8 \times 5) + (8 \times 2) = 40 + 16 = 56$. (Distributive property.)

- Apply and extend previous understandings of division to divide unit fractions by whole numbers and whole numbers by unit fractions. Students able to multiply fractions in general can develop strategies to divide fractions in general, by reasoning about the relationship between multiplication and division.

Addressing

- Use ratio and rate reasoning to solve real-world and mathematical problems, e.g., by reasoning about tables of equivalent ratios, bar models, double number line diagrams, or equations.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- Group Presentations
- Clarify, Critique, Correct
- Discussion Supports
- True or False

Required Materials

Graph paper
Multi-link cubes
Tools for creating a visual display
Any way for students to create work that can be easily displayed to the class. Examples: chart paper and markers, whiteboard space and markers, shared online drawing tool, access to a document camera.

Required Preparation

Prepare a set of 50 red multi-link cubes and 30 blue multi-link cubes for each group of students.

Student Learning Goals

Let’s look at situations where you can add the quantities in a ratio together.

15.1 True or False: Multiplying by a Unit Fraction

Warm Up: 10 minutes

This warm-up encourages students to use the meaning of fractions and properties of operations to reason about equations. While students may evaluate each side of the equation to determine if it is true or false, encourage students to think about the following ideas in each:

- The first question: Dividing is the same as multiplying by the reciprocal of the divisor.

- The second question: Adjusting the factors adjusts the products. If both factors increase, the resulting product will be greater than the original.
- The third question: The commutative property of multiplication.
- The fourth question: Decomposing a dividend into two numbers and dividing each by the divisor is a way to find the quotient of the original dividend.

Instructional Routines

- True or False

Launch

Display one problem at a time. Tell students to give a signal when they have an answer and a strategy. After each problem, give students 1 minute of quiet think time and follow with a whole-class discussion.

Student Task Statement

True or false?

$$\frac{1}{5} \times 45 = \frac{45}{5}$$

$$\frac{1}{5} \times 20 = \frac{1}{4} \times 24$$

$$42 \times \frac{1}{6} = \frac{1}{6} \times 42$$

$$486 \times \frac{1}{12} = \frac{480}{12} + \frac{6}{12}$$

Student Response

1. True. Division is the same as multiplying by the reciprocal.
2. False. Both factors increased.
3. True. Commutative property of multiplication.
4. True. Partial quotients.

Activity Synthesis

Ask students to share their strategies for each problem. Record and display their explanations for all to see. Ask students if or how the factors in the problem impacted the strategy choice. To involve more students in the conversation, consider asking:

- "Who can restate ___'s reasoning in a different way?"
- "Does anyone want to add on to _____'s strategy?"

- “Do you agree or disagree? Why?”

After each true equation, ask students if they could rely on the reasoning used on the given problem to think about or solve other problems that are similar in type. After each false equation, ask students how we could make the equation true.

15.2 Cubes of Paint

10 minutes

Up until now, students have worked with ratios of quantities given in terms of specific units such as millilitres, cups, teaspoons, etc. This task introduces students to the use of the more generic “parts” as a unit in ratios, and the use of **bar models** to represent such ratios. In addition to thinking about the ratio between two quantities, students also begin to think about the ratio between the two quantities and their total.

Two important ideas to make explicit through the task and discussion:

- A ratio can associate quantities given in terms of a specific unit (as in 4 teaspoons of this to 3 teaspoons of that). A ratio can also associate quantities of the same kind without specifying particular units, in terms of “parts” (as in 4 parts of this to 3 parts of that). Any appropriate unit can be used in place of “parts” without changing the 4 to 3 ratio.
- A ratio can tell us about how two or more quantities relate to one another, but it can also tell us about the combined quantity (when that makes sense) and allow us to solve problems.

As students work, notice in particular how they approach the last two questions. Identify students who add multi-link cubes to represent the larger amount of paint, and those who use the original number of multi-link cubes but adjust their reasoning about what each cube represents. Be sure to leave enough time to debrief as a class and introduce bar models afterwards.

Instructional Routines

- Discussion Supports

Launch

Explain to students that they will explore paint mixtures and use multi-link cubes to represent them. Say: “To make a particular green paint, we need to mix 1 ml of blue paint to 3 ml of yellow.” Represent this recipe with 1 blue multi-link cube and 3 yellow ones and display each set horizontally (to mimic the appearance of a bar model).

Ask:

- "How much green paint will this recipe yield?" (4 ml of green paint.)
- "If each cube represents 2 ml instead of 1 ml, how much of blue and yellow do the multi-link cubes represent? How many ml of green paint will we have?" (2 ml of blue, 6 ml of yellow, and 8 ml of green.)
- "Is there another way to represent 2 ml of blue and 6 ml of yellow using multi-link cubes?" (We could use 2 blue multi-link cubes and 6 yellow ones.)
- "How do we refer to 2 ml of blue and 6 ml of yellow in terms of 'batches'?" (2 batches.)

Highlight the fact that they could either represent 2 ml of blue and 6 ml of yellow with 2 blue multi-link cubes and 6 yellow ones (show this representation, if possible), or with 1 blue multi-link cube and 3 yellow ones (show representation), with the understanding that each cube stands for 2 ml of paint instead of 1 ml.

Explain to students that, in the past, they had thought about different amounts of ingredients in a recipe in terms of batches, but in this task they will look at another way to mix the right amounts specified by a ratio.

Arrange students in groups of 3–5. Provide 50 red multi-link cubes and 30 blue multi-link cubes to each group. Give groups time to complete the activity, and then debrief as a class.

Representation: Develop Language and Symbols. Use virtual or concrete manipulatives to connect symbols to concrete objects or values. Provide students with multi-link cubes, blocks or printed representations.
Supports accessibility for: Conceptual processing

Anticipated Misconceptions

Students may need help interpreting "Suppose each cube represents 2 ml." If necessary, suggest they keep using one cube to represent 1 ml of paint. So, for example, the second question would be represented by 5 *stacks of 2* red cubes and 3 *stacks of 2* blue cubes. If they use that strategy, each part of the bar model would represent one stack.

Student Task Statement

A recipe for maroon paint says, "Mix 5 ml of red paint with 3 ml of blue paint."

1. Use multi-link cubes to represent the amounts of red and blue paint in the recipe. Then, draw a sketch of your multi-link cube representation of the maroon paint.

   a. What amount does each cube represent?

   b. How many millilitres of maroon paint will there be?

   a. Suppose each cube represents 2 ml. How much of each colour paint is there?

   Red: ______ ml

   Blue: ______ ml

   Maroon: ______ ml

   b. Suppose each cube represents 5 ml. How much of each colour paint is there?

   Red: ______ ml

   Blue: ______ ml

   Maroon: ______ ml

   a. Suppose you need 80 ml of maroon paint. How much red and blue paint would you mix? Be prepared to explain your reasoning.

   Red: ______ ml

   Blue: ______ ml

   Maroon: 80 ml

   b. If the original recipe is for one batch of maroon paint, how many batches are in 80 ml of maroon paint?

Student Response

1. Show 5 red multi-link cubes and 3 blue ones.

   a. Each multi-link cube represents 1 ml.

   b. $1 + 1 + 1 + 1 + 1 = 5$, so there is 5 ml of red paint. $1 + 1 + 1 = 3$, so there is 3 ml of blue paint. $5 + 3 = 8$, so there is 8 ml of maroon paint.

   a. $2 + 2 + 2 + 2 + 2 = 10$, so there is 10 ml of red paint. $2 + 2 + 2 = 6$, so there is 6 ml of blue paint. $10 + 6 = 16$, so there is 16 ml of maroon paint.

   b. There is 25 ml of red, since $5 \times 5 = 25$, and 15 ml of blue, since $5 \times 3 = 15$. $25 + 15 = 40$, so there is 40 ml of maroon paint.

   a. $80 \div 8 = 10$ and $10 \times 5 = 50$, so there is 50 ml red. $10 \times 3 = 30$, so there is 30 ml blue. $50 + 30 = 80$, so there is 80 ml maroon.

b. There are 10 batches of paint, because each part changed from a value 1 ml to a value of 10 ml.

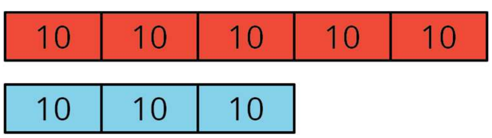
Activity Synthesis

Class discussion should centre around how students used multi-link cubes to answer the questions and their approach to the last two questions. Invite some students to share their group's approach. Ask:

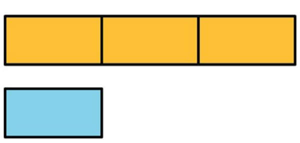
- "How did the multi-link cubes help you solve the first few problems?"
- "In one of the problems, you were only given the total amount of maroon paint. How did you find out the amounts of blue and red paint needed to produce 80 ml of maroon?"
- "How did you approach the last question?" (Add more cubes, or use the same representation of 5 red cubes and 3 blue ones.)

Discuss how the same 5 red cubes and 3 blue ones can be used to represent a total of 80 ml of blue paint. Explain that this situation can be represented with a **bar model**. A **bar model** is a horizontal strip that is partitioned into parts. Each part (like each multi-link cube) represents a value. It can be any value, as long as the same value is used throughout.

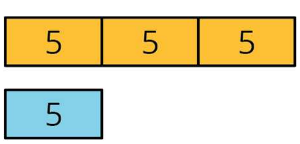
Show a bar model representing a 5 : 3 ratio of red paint to blue paint yielding 80 ml of maroon paint. Ask students where they see the 5, the 3, and the 80 being represented in the diagram. Discuss how many batches of paint are represented.



Show the bar model for green paint mixture discussed earlier. Students should be able to say that the ratio of blue to yellow paint is 1 : 3. Ask: "What value each part of the diagram would have to take to show a 20 ml mixture of green paint? How do you know?"



Guide students to see that, if each of the 4 total parts must be equal in value and amount to 20 ml, we could divide 20 by 5 to find out what each part represents. $20 \div 4 = 5$, so each part represents 5 ml of paint.



Representing: Discussion Supports. To help students connect ratio language and ratio reasoning, invite a student to represent their reasoning using the multi-link cubes or with a bar model. Press for details by requesting that students challenge an idea, elaborate on an idea, or give an example. This will help students communicate with precise language. *Design Principle(s): Support sense-making*

15.3 Trainers, Chicken, and Fruit Juice

20 minutes

This activity allows students to practise reasoning about situations involving ratios of two quantities and their sum. It also introduces students to using “parts” in recipes (e.g., 3 parts oil with 2 parts soy sauce and 1 part orange juice), instead of more familiar units such as cups, teaspoons, millilitres, etc. Students may use bar models to support their reasoning, or they may use other representations learned so far—discrete diagrams, number lines, tables, or equations. All approaches are welcome as long as students use them to represent the situations appropriately to support their reasoning.

As students work, monitor for different ways students reason about the problems, with or without using bar models.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect

Launch

Keep students in the same groups. Provide graph paper and multi-link cubes (any three colours). Explain that they will now practise solving problems involving ratios and their combined quantities (similar to the green and purple paint in the previous task). Draw students to a ratio that uses “parts” as its unit. Ask students what they think “one part” means or amounts to, and how situations expressed in terms of “parts” could be diagrammed.

Before students begin working, make sure they understand that “parts” do not represent specific amounts, that the value of “one part” can vary but the size of all parts is equal, and that a bar model can be used to show these parts.

Anticipated Misconceptions

Students may think of each segment of a bar model as representing each cube, rather than as a flexible representation of an increment of a quantity. Help them set up the bars with the correct number of sections and then discuss how many parts there are in all.

Student Task Statement

Solve each of the following problems and show your thinking. If you get stuck, consider drawing a **bar model** to represent the situation.

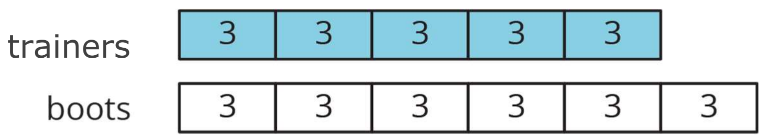
1. The ratio of students wearing trainers to those wearing boots is 5 to 6. If there are 33 students in the class, and all of them are wearing either trainers or boots, how many of them are wearing trainers?

2. A recipe for chicken marinade says, "Mix 3 parts oil with 2 parts soy sauce and 1 part orange juice." If you need 42 cups of marinade in all, how much of each ingredient should you use?

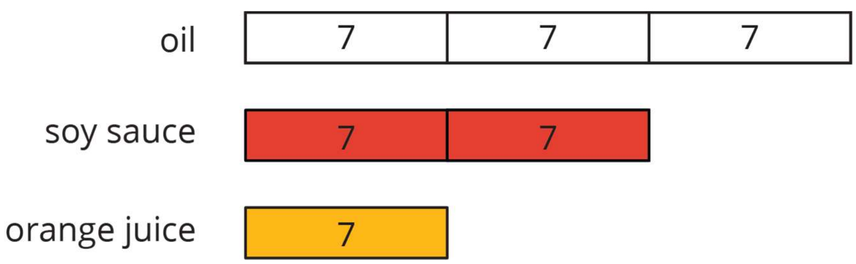
3. Elena makes fruit punch by mixing 4 parts cranberry juice to 3 parts apple juice to 2 parts grape juice. If one batch of fruit punch includes 30 cups of apple juice, how large is this batch of fruit punch?

Student Response

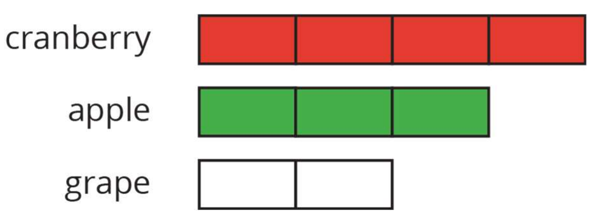
1. 15 students are wearing trainers. $33 \div 11 = 3$. The value of each unit is 3. $3 \times 5 = 15$.



2. 14 cups of soy sauce. $42 \div 6 = 7$. The value of each unit is 7. $7 \times 3 = 21$. There are 21 cups of oil. $7 \times 2 = 14$. There are 14 cups of soy sauce. $1 \times 7 = 7$. There are 7 cups of orange juice.



3. 90 cups of punch. $30 \div 3 = 10$. The value of each unit is 10. $10 \times 3 = 30$. There are 30 cups of apple juice. $10 \times 4 = 40$. There are 40 cups cranberry juice. $10 \times 2 = 20$. There are 20 cups grape juice. $40 + 30 + 20 = 90$.



Are You Ready for More?

Using the recipe from earlier, how much fruit punch can you make if you have 50 cups of cranberry juice, 40 cups of apple juice, and 30 cups of grape juice?

Student Response

$112\frac{1}{2}$ cups. Figure out which ingredient you'd run out of first. Cranberry juice limits you to $12\frac{1}{2}$ batches, less than $13\frac{1}{3}$ for apple juice and 15 for grape juice. Then, multiply by 9 cups per batch.

Activity Synthesis

Select students to share their reasoning. Help students make connections between different representations, especially any bar models.

15.4 Invent Your Own Ratio Problem

Optional: 10 minutes

In this activity, students have an opportunity to create their own equivalent ratio problem.

Instructional Routines

- Group Presentations
- Clarify, Critique, Correct

Launch

Keep students in the same groups. Provide graph paper, multi-link cubes (any three colours), and tools for creating a visual display.

Engagement: Internalise Self-Regulation. Check for understanding by inviting students to rephrase directions in their own words. Provide a project checklist that chunks the various steps of the activity into a set of manageable tasks.
Supports accessibility for: Organisation; Attention

Student Task Statement

1. Invent another ratio problem that can be solved with a bar model and solve it. If you get stuck, consider looking back at the problems you solved in the earlier activity.
2. Create a visual display that includes:
   - The new problem that you wrote, without the solution.
   - Enough work space for someone to show a solution.
3. Swap your display with another group, and solve each other's problem. Include a bar model as part of your solution. Be prepared to share the solution with the class.
4. When the solution to the problem you invented is being shared by another group, check their answer for accuracy.

Student Response

Answers vary.

Activity Synthesis

Have each group share the peer-generated question it was assigned and the solution. Though the group that wrote the question will be responsible for confirming the answer, encourage all to listen to the reasoning each group used.

Reading, Writing: Clarify, Critique, Correct. Use this routine to provide students with the opportunity to consider the important details and language that should be included in a ratio problem. Ask students to think about what the ratio problems they solved in the earlier activity all had in common, then display the following problem, "There are 5 lions and 2 birds. If there are 20 animals in the zoo, how many are lions or birds?" Give students 2 minutes of quiet think time to consider what is missing or unclear about the problem. Prompt discussion by asking, "What can we change to make this a better ratio problem?" Call students' attention to the language used to communicate the information necessary to solve a ratio problem, and to the importance of values that make sense for a given situation. If time allows, invite students to write and share a revised version of this problem.
Design Principle(s): Maximise meta-awareness; Optimise output (for explanation)

Lesson Synthesis

Today's ratio problems were different from the ones we've worked on so far because they include an additional piece of information:

- Can anyone identify what made these problems different? (They include the combined or total amount of the quantities in the ratio. This is possible because in each problem there was only one unit of measure and the total of the quantities made sense in the context.)
- How can a **bar model** represent these types of situations? (Each part of the bar represents a particular value, and the sum of those values represents the total amount.)
- How does changing the value of each part of the bar affect the total amount? (If the value is different, the combined sum will be different.) Review the use of a bar model for representing and solving a problem involving the total amount.

15.5 Room Sizes

Cool Down: 5 minutes

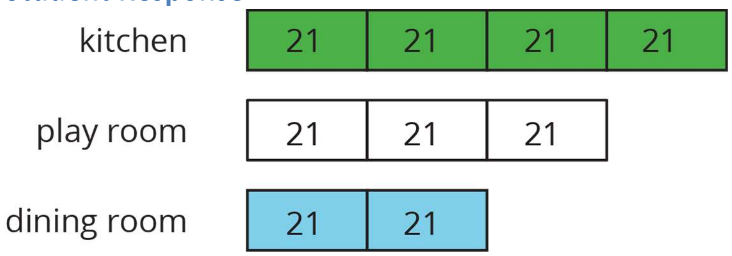
Launch

Provide access to graph paper.

Student Task Statement

The first floor of a house consists of a kitchen, playroom, and dining room. The areas of the kitchen, playroom, and dining room are in the ratio 4 : 3 : 2. The combined area of these three rooms is 189 square feet. What is the area of each room?

Student Response

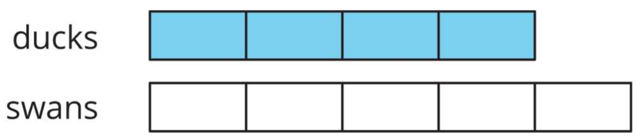


All three rooms amount to 9 units. All three rooms make 189 square feet. $189 \div 9 = 21$, so each part of the bar model represents 21 square feet. The area of the kitchen is 84 square feet, the area of the playroom is 63 square feet, and the area of the dining room is 42 square feet.

Student Lesson Summary

A **bar model** is another way to represent a ratio. All the parts of the diagram that are the same size have the same value.

For example, this bar model represents the ratio of ducks to swans in a pond, which is $4 : 5$.

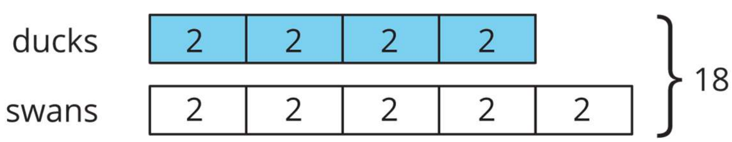


The first bar represents the number of ducks. It has 4 parts.

The second bar represents the number of swans. It has 5 parts.

There are 9 parts in all, because $4 + 5 = 9$.

Suppose we know there are 18 of these birds in the pond, and we want to know how many are ducks.



The 9 equal parts on the diagram need to represent 18 birds in all. This means that each part of the bar model represents 2 birds, because $18 \div 9 = 2$.

There are 4 parts of the bar representing ducks, and $4 \times 2 = 8$, so there are 8 ducks in the pond.

Glossary

- bar model

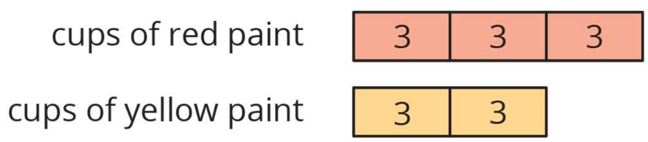
Lesson 15 Practice Problems

Problem 1 Statement

Here is a bar model representing the ratio of red paint to yellow paint in a mixture of orange paint.

a. What is the ratio of yellow paint to red paint?

b. How many total cups of orange paint will this mixture yield?



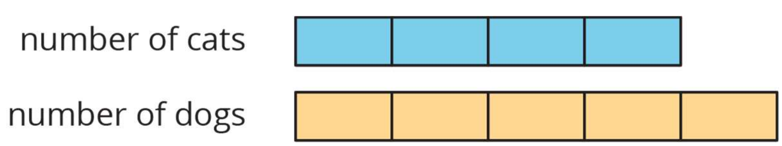
Solution

a. 2 : 3 (or equivalent)

b. 15 cups

Problem 2 Statement

At the kennel, the ratio of cats to dogs is 4 : 5. There are 27 animals in all. Here is a bar model representing this ratio.



a. What is the value of each small rectangle?

b. How many dogs are at the kennel?

c. How many cats are at the kennel?

Solution

a. Each unit is 3, because $4 + 5 = 9$ and $27 \div 9 = 3$.

b. There are 15 dogs, because $3 \times 5 = 15$.

c. There are 12 cats, because $3 \times 4 = 12$.

Problem 3 Statement

Last month, there were 4 sunny days for every rainy day. If there were 30 days in the month, how many days were rainy? Explain your reasoning. If you get stuck, consider using a bar model.

Solution

There were 6 rainy days, because $4 + 1 = 5$, so there are 5 units total. $30 \div 5 = 6$, so each unit is worth 6.

Problem 4 Statement

Noah entered a 100-mile cycle race. He knows he can ride 32 miles in 160 minutes. At this rate, how long will it take him to finish the race? Use each table to find the answer. Next, explain which table you think works better in finding the answer.

Table A:

distance (miles)	elapsed time (minutes)
32	160
1	
100	

Table B:

distance (miles)	elapsed time (minutes)
32	160
96	
4	
100	

Solution

He will finish the race in 500 minutes (or equivalent).

Table A:

distance (miles)	elapsed time (minutes)
32	160
1	5
100	500

Table B:

distance (miles)	elapsed time (minutes)
32	160
96	480

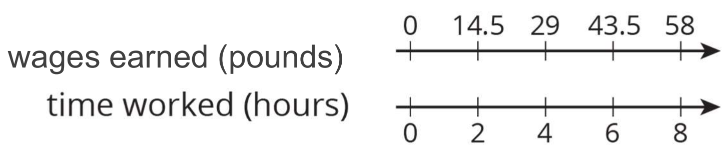
4	20
100	500

Answers vary. Sample response: The first table is more efficient, but they both work in getting the answer.

Problem 5 Statement

A cashier worked an 8-hour day, and earned £58.00. For each question, explain your reasoning.

The double number line shows the amount she earned for working different numbers of hours.



a. How much does the cashier earn per hour?

b. How much does the cashier earn if she works 3 hours?

Solution

a. £7.25 per hour. Possible reasoning: $14.5 \div 2 = 7.25$

b. £21.75. Possible reasoning: $(7.25) \times 3 = 21.75$

Problem 6 Statement

A supermarket store sells bags of oranges in two different sizes.

– The 3-pound bags of oranges cost £4.

– The 8-pound bags of oranges for £9.

Which oranges cost less per pound? Explain or show your reasoning.

Solution

The 8-pound bags cost less per pound. Possible strategies:

– Compare the cost for 24 pounds of oranges for both types of bags. 24 pounds cost £32 when sold in 3-pound bags. 24 pounds cost £27 when sold in 8-pound bags.

– Compare how much can be bought for the same amount of money. £36 can buy 27 pounds of oranges in 3-pound bags, or it can buy 32 pounds in 8-pound bags.